

Supersymmetric Gauge Theories with Flavors and Matrix Models

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Abstract

We present two results concerning the relation between poles and cuts by using the example of $\mathcal{N} = 1$ $U(N_c)$ gauge theories with matter fields in the adjoint, fundamental and anti-fundamental representations. The first result is the on-shell possibility of poles, which are associated with flavors and on the second sheet of the Riemann surface, passing through the branch cut and getting to the first sheet. The second result is the generalization of hep-th/0311181 (Intriligator, Kraus, Ryzhov, Shigemori, and Vafa) to include flavors. We clarify when there are closed cuts and how to reproduce the results of the strong coupling analysis by matrix model, by setting the glueball field to zero from the beginning. We also make remarks on the possible stringy explanations of the results and on generalization to $SO(N_c)$ and $USp(2N_c)$ gauge groups.

1 Introduction

String theory can be a powerful tool to understand four dimensional supersymmetric gauge theory which exhibits rich dynamics and allows an exact analysis. In [1], using the generalized Konishi anomaly and matrix model [2], $\mathcal{N} = 1$ supersymmetric $U(N_c)$ gauge theory with matter fields in the adjoint, fundamental and anti-fundamental representations was studied. The resolvents in the quantum theory live on the two-sheeted Riemann surface defined by the matrix model curve. Their quantum behavior is characterized by the structure around the branch cuts and poles, which are related to the RR flux contributions in the Calabi–Yau geometry and flavor fields, respectively. A pole associated with flavor on the first sheet is related to the Higgs vacua (corresponding to classical nonzero vacuum expectation value of the fundamental) while a pole on the second sheet is related to the pseudo-confining vacua where the classically vanishing vacuum expectation value of the fundamental gets nonzero values due to quantum correction.

It is known [1] that Higgs vacua and pseudo-confining vacua, which are distinct in the classical theory, are smoothly transformed into each other in the quantum theory. This transition is realized on the Riemann surface by moving poles located on the second sheet to pass the branch cuts and enter the first sheet. This process was analyzed in [1] at the off-shell level by fixing the value of glueball fields during the whole process. However, in an on-shell process, the position of poles and the width and position of branch cuts are correlated (when the flavor poles are moved, the glueball field is also changed). It was conjectured in [1] that for a given branch cut, there is an upper bound for the number of poles (the number of flavors) which can pass through the cut from the second sheet to the first sheet.

Our first aim of this paper is to confirm this conjecture and give the corresponding upper bound for various gauge groups (in particular, we will concentrate on the $U(N_c)$ gauge group). The main result is that if $N_f \geq N_c$, the poles will not be able to pass through the cut to the first sheet where N_c is the effective fluxes associated with the cut (and can be generalized to other gauge groups).

Another important development was made in [3], which was inspired by [4]. In [3], which we will refer to as IKRSV, it was shown that, to correctly compute the prediction of string theory (matrix model), it is crucial to determine whether the glueball is really a good variable or not. A prescription was given, regarding when a glueball field corresponding to a given branch cut should be set to zero before extremizing the off-shell glueball superpotential. The discussion of IKRSV was restricted to $\mathcal{N} = 1$ gauge theories with an adjoint and no flavors, so the generalization to the case with fundamental flavors is obviously the next task.

Our second aim of this paper is to carry out this task. The main result is the following. Assuming N_f poles around a cut associated with gauge group $U(N_{c,i})$, when $N_f \geq N_{c,i}$ there are situations in which we should set $S_i = 0$ in matrix model computations. More concretely, situations with $S_i = 0$ belong to either of the following two branches: the baryonic branch for $N_{c,i} \leq N_f < 2N_{c,i}$, or the $r = N_{c,i}$ non-baryonic branch for $N_f \geq 2N_{c,i}$. Moreover, when $S_i = 0$, the gauge group is completely broken and there should exist some extra, charged massless field which is not incorporated in matrix model.

In section 2, as background, we review basic materials for $\mathcal{N} = 1$ supersymmetric $U(N_c)$ gauge theory with an adjoint chiral superfield, and N_f flavors of quarks and anti-quarks. The chiral operators and the exact effective glueball superpotential are given. We study the vacuum structure of the gauge theory at classical and quantum levels. We review also the main results of IKRSV. In addition to all these reviews, we present our main motivations of this paper.

In section 3, we apply the formula for the off-shell superpotential obtained in [1] to the case with quadratic tree level superpotential, and solve the equation of motion derived from it. We consider what happens if one moves N_f poles associated with flavors on the second sheet through the cut onto the first sheet, on-shell. Also, in subsection 3.4, we briefly touch the matter of generalizing IKRSV in the one cut model.

In section 4, we consider cubic tree level superpotential. On the gauge theory side, the factorization of the Seiberg–Witten curve provides an exact superpotential. We reproduce this superpotential by matrix model, by extremizing the effective glueball superpotential with respect to glueball fields after setting the glueball field to zero when necessary. We present explicit results for $U(3)$ theory with all possible breaking patterns and different number of flavors ($N_f = 1, 2, 3, 4$, and 5).

In section 5, after giving concluding remarks, we repeat the procedure we did in previous sections for $SO(N_c)/USp(2N_c)$ theories, briefly.

In the appendix, we present some proofs and detailed calculations which are necessary for the analysis in section 4.

Since string theory results in the dual Calabi–Yau geometry are equivalent to the matrix model results, we refer to them synonymously through the paper. There exist many related works to the present paper. For a list of references, we refer the reader to [5].

2 Background

In this section, we will summarize the relevant background needed for the study of $\mathcal{N} = 1$

supersymmetric gauge theory with matter fields.

2.1 The general picture of matrix model with flavors

The generalized Konishi anomaly interpretation to the matrix model approach for $\mathcal{N} = 1$ supersymmetric gauge theory with flavors was given in [6, 1]. Here we make only a brief summary on some points we will need.

Let us consider $\mathcal{N} = 1$ supersymmetric $U(N_c)$ gauge theory, coupled to an adjoint chiral superfield Φ , N_f fundamentals Q^f , and N_f anti-fundamentals $\tilde{Q}_{\tilde{f}}$. The tree level superpotential is taken to be

$$W_{\text{tree}} = \text{Tr } W(\Phi) + \sum_{f, \tilde{f}} \tilde{Q}_{\tilde{f}} m_{\tilde{f}}^f(\Phi) Q^f, \quad (2.1)$$

where the function $W(z)$ and the matrix $m_{\tilde{f}}^f(z)$ are polynomials

$$W(z) = \sum_{k=0}^n \frac{g_k z^{k+1}}{k+1}, \quad m_{\tilde{f}}^f(z) = \sum_{k=1}^{l+1} (m_k)^{\tilde{f}}_f z^{k-1}.$$

Classically we can have the “pseudo-confining vacua” where the vacuum expectation values of Q , \tilde{Q} are zero, or the “Higgs vacua” where the vacuum expectation values of Q , \tilde{Q} are nonzero so that the total rank of the remaining gauge groups is reduced. These two vacua, which seem to have a big difference classically, are not fundamentally distinguishable from each other in the quantum theory and in fact can be continuously transformed into each other, as we will review shortly, in the presence of flavors [1].

Supersymmetric vacua of gauge theory are characterized by the vacuum expectation values of chiral operators [7]. They are nicely packaged into the following functions called resolvents [1, 6]:¹

$$T(z) = \left\langle \text{Tr} \frac{1}{z - \Phi} \right\rangle, \quad (2.2)$$

$$R(z) = -\frac{1}{32\pi^2} \left\langle \text{Tr} \frac{W_\alpha W^\alpha}{z - \Phi} \right\rangle, \quad (2.3)$$

$$M(z)^f_{\tilde{f}} = \left\langle \tilde{Q}_{\tilde{f}} \frac{1}{z - \Phi} Q^f \right\rangle \quad (2.4)$$

where W_α is (the lowest component of) the field strength superfield. Classically, $R(z)$ vanishes while $T(z)$, $M(z)$ have simple poles on the complex z -plane at infinity and at the eigenvalues

¹We set $w_\alpha(z) \equiv \frac{1}{4\pi} \left\langle \text{Tr} \frac{W_\alpha}{z - \Phi} \right\rangle$ to zero because in supersymmetric vacua $w_\alpha(z) = 0$.

of Φ . Each eigenvalue of Φ is equal to one of zeros of $W'(z)$ or $B(z)$, where

$$W'(z) = g_n \prod_{i=1}^n (z - a_i), \quad B(z) \equiv \det m(z) = B_L \prod_{I=1}^L (z - z_I). \quad (2.5)$$

In the pseudo-confining vacuum, every eigenvalue of Φ is equal to a_i for some i . On the other hand, in the Higgs vacuum, some eigenvalues of Φ are equal to z_I for some I .

In the quantum theory, the resolvents (2.2), (2.3) and (2.4) are determined by the generalized Konishi anomaly equations [8, 6, 1]:

$$\begin{aligned} [W'(z)T(z)]_- + \text{Tr}[m'(z)M(z)]_- &= 2R(z)T(z), \\ [W'(z)R(z)]_- &= R(z)^2, \\ [(M(z)m(z))^{f'}_-]_- &= R(z)\delta_f^{f'}, \\ [(m(z)M(z))^{\tilde{f}'}_-]_- &= R(z)\delta_{\tilde{f}}^{\tilde{f}'}, \end{aligned} \quad (2.6)$$

where the notation $[]_-$ means to drop the nonnegative powers in a Laurent expansion in z . From the second equation of (2.6), one obtains [9]

$$R(z) = \frac{1}{2} \left(W'(z) - \sqrt{W'(z)^2 + f(z)} \right),$$

where $f(z)$ is a polynomial of degree $(n-1)$ in z . This implies that in the quantum theory the zeros $z = a_i$ ($i = 1, 2, \dots, n$) of $W'(z)$ are blown up into cuts A_i along intervals² $[a_i^-, a_i^+]$ by the quantum effect represented by $f(z)$, and the resolvents (2.2)–(2.4) are defined on a double cover of the complex z -plane branched at the roots a_i^\pm of $W'(z)^2 + f(z)$. This double cover of the z -plane can be thought of as a Riemann surface Σ described by the matrix model curve

$$\Sigma : \quad y_m^2 = W'(z)^2 + f(z). \quad (2.7)$$

This curve is closely related to the factorization form of $\mathcal{N} = 2$ curve in the strong coupling analysis.

Every point z on the z -plane is lifted to two points on the Riemann surface Σ which we denote by q and \tilde{q} respectively. For example, z_I is lifted to q_I on the first sheet and \tilde{q}_I on the second sheet. We write the projection from Σ to the z -plane as $z_I = z(q_I) = z(\tilde{q}_I)$, following the notation of [1].

The classical singularities of the resolvents $T(z)$, $M(z)$ are modified in the quantum theory to the singularities on Σ , as follows. For $T(z)$, the classical poles at z_I are lifted to poles at q_I

² a_i^\pm are generally complex and in such cases we take A_i to be a straight line connecting a_i^- and a_i^+ . Note that there is no physical meaning to the choice of the cut; it can be any path connecting a_i^- and a_i^+ .

or \tilde{q}_I , depending on which vacuum the theory is in, while the classical poles at a_i with residue $N_{c,i}$ are replaced by cuts with periods $\frac{1}{2\pi i} \oint_{A_i} T(z)dz = N_{c,i}$. For $M(z)$, the classical poles at z_I are also lifted to poles at q_I or \tilde{q}_I . More specifically, by solving the last two equations of (2.6), one can show [1]

$$M(z) = R(z) \frac{1}{m(z)} - \sum_{I=1}^L \frac{(1-r_I)R(q_I)}{(z-z_I)} \frac{1}{2\pi i} \oint_{q_I} \frac{dx}{m(x)} - \sum_{I=1}^L \frac{r_I R(\tilde{q}_I)}{(z-z_I)} \frac{1}{2\pi i} \oint_{\tilde{q}_I} \frac{dx}{m(x)}, \quad (2.8)$$

where (q_I, \tilde{q}_I) are the lift of z_I to the first sheet and to the second sheet of Σ , and $r_I = 0$ for poles on the second sheet and $r_I = 1$ for poles on the first sheet. Furthermore, for $T(z)$, by solving the first equation of (2.6),

$$T(z) = \frac{B'(z)}{2B(z)} - \sum_{I=1}^L \frac{(1-2r_I)y(q_I)}{2y(z)(z-z_I)} + \frac{c(z)}{y(z)}, \quad (2.9)$$

where

$$c(z) = \left\langle \text{Tr} \frac{W'(z) - W'(\Phi)}{z - \Phi} \right\rangle - \frac{1}{2} \sum_{I=1}^L \frac{W'(z) - W'(z_I)}{z - z_I}. \quad (2.10)$$

Practically it is hard to use (2.10) to obtain $c(z)$ and we use the following condition instead:

$$\frac{1}{2\pi i} \oint_{A_i} T(z)dz = N_{c,i}. \quad (2.11)$$

Finally, the exact, effective glueball superpotential is given by [1]

$$\begin{aligned} W_{\text{eff}} = & -\frac{1}{2} \sum_{i=1}^n N_{c,i} \int_{\hat{B}_i^r} y(z)dz - \frac{1}{2} \sum_{I=1}^L (1-r_I) \int_{\tilde{q}_I}^{\tilde{\Lambda}_0} y(z)dz - \frac{1}{2} \sum_{I=1}^L r_I \int_{q_I}^{\tilde{\Lambda}_0} y(z)dz \\ & + \frac{1}{2} (2N_c - L) W(\Lambda_0) + \frac{1}{2} \sum_{I=1}^L W(z_I) - \pi i (2N_c - L) S + 2\pi i \tau_0 S + 2\pi i \sum_{i=1}^{n-1} b_i S_i, \end{aligned} \quad (2.12)$$

where

$$2\pi i \tau_0 = \log \left(\frac{B_L \Lambda^{2N_c - N_f}}{\Lambda_0^{2N_c - L}} \right). \quad (2.13)$$

Here, Λ_0 is the cut-off of the contour integrals, Λ is the dynamical scale, $S \equiv \sum_{i=1}^n S_i$, and $b_i \in \mathbb{Z}$. \hat{B}_i^r is the regularized contour from $\tilde{\Lambda}_0$ to Λ_0 through the i -th cut and Λ_0 and $\tilde{\Lambda}_0$ are the points on the first sheet and on the second sheet, respectively. The glueball field is defined as

$$S_i = \frac{1}{2\pi i} \oint_{A_i} R(z)dz.$$

In the above general solutions (2.8), (2.9), we have $r_I = 1$ or $r_I = 0$, depending on whether the pole is on the first sheet or on the second sheet of Σ , respectively. The relation between these choices of r_I and the phase of the system is as follows. Let us start with all $r_I = 0$, i.e., all the poles are on the second sheet. This choice corresponds to the pseudo-confining vacua where the gauge group is broken as $U(N_c) \rightarrow \prod_{i=1}^n U(N_{c,i})$ with $\sum_{i=1}^n N_{c,i} = N_c$. Now let us move a single pole through, for example, the n -th cut to the first sheet. This will break the gauge group as $\prod_{i=1}^n U(N_{c,i}) \rightarrow \prod_{i=1}^{n-1} U(N_{c,i}) \times U(N_{c,n} - 1)$. Note that the rank of the last factor is now $(N_{c,n} - 1)$ so that $\sum_{i=1}^n N_{c,i} = N_c - 1 < N_c$. Namely, the gauge group is Higgsed down. In this way, by passing poles through cuts, one can go continuously from the pseudo-confining phase to the Higgs phase, as advocated before.

However, if we consider this process of passing poles through a cut to the first sheet *on-shell*, then there should be an obstacle at a certain point. For example, if initially we have $N_{c,n} = 1$, after passing a pole we would end up with an $U(0)$. This sudden jump of the number of $U(1)$'s in the low energy gauge theory is not a smooth physical process, because the number of massless particles (photons) changes discontinuously. So we expect some modifications to the above picture. In [1], it was suggested that in an on-shell process, the n -th cut will close up in such a situation so that the pole cannot pass through. It is one of our motivations to show that this is indeed true. More precisely, the cut does not close up completely and the pole can go through a little bit further to the first sheet and then will be bounced back to the second sheet.

2.2 The vacuum structure

In the last subsection we saw that different distributions of poles over the first and the second sheets correspond to different phases of the theory. In this subsection we will try to understand this vacuum structure of the gauge theory at both classical and quantum levels for a specific model (for more details, see [10, 11, 12, 13, 14, 15]). For simplicity we will focus on $U(N_c)$ theory with N_f flavors and the following tree level superpotential ³

$$W_{\text{tree}} = \frac{1}{2} m_A \text{Tr } \Phi^2 - \sum_{I=1}^{N_f} \tilde{Q}_I (\Phi + m_f) Q^I. \quad (2.14)$$

This corresponds to taking polynomials in (2.1) as

$$W(z) = \frac{m_A}{2} z^2, \quad m_{\tilde{I}}^{\tilde{I}}(z) = -(z + m_f) \delta_{\tilde{I}}^{\tilde{I}}.$$

³We used the convention of [16] for the normalization of the second term. Different choices are related to each other by redefinition of \tilde{Q} and Q .

All N_f flavors have the same mass m_f , and the mass function defined in (2.5) is given by

$$B(z) = (-1)^{N_f} (z + m_f)^{N_f}.$$

Therefore, poles associated with flavors are located at

$$z_I = -m_f \equiv z_f, \quad I = 1, 2, \dots, N_f. \quad (2.15)$$

In the quantum theory, some of these poles are lifted to q_f on the first sheet and others are lifted to \tilde{q}_f on the second sheet.

The D - and F -flatness for the superpotential (2.14) is given by

$$0 = [\Phi, \Phi^\dagger], \quad 0 = QQ^\dagger - \tilde{Q}^\dagger \tilde{Q}, \quad 0 = m_A \Phi - Q \tilde{Q}, \quad 0 = (\Phi + m_f)Q = \tilde{Q}(\Phi + m_f).$$

Solutions are a little different for $m_f \neq 0$ and $m_f = 0$, because $m_f = 0$ is the root of $W'(z) = z$. The case of $W'(-m_f) = 0$ was discussed in [10, 13] which we will refer to as the classically massless case.

In the $m_f \neq 0$ case, the solution is given by

$$\begin{aligned} \Phi &= \begin{bmatrix} -m_f I_{K \times K} & 0 \\ 0 & 0_{(N_c-K) \times (N_c-K)} \end{bmatrix}, \\ Q &= \begin{bmatrix} A_{K \times K} & 0 \\ 0 & 0_{(N_c-K) \times (N_f-K)} \end{bmatrix}, \quad {}^t \tilde{Q} = \begin{bmatrix} {}^t \tilde{A}_{K \times K} & \tilde{B}_{K \times (N_f-K)} \\ 0 & 0_{(N_c-K) \times (N_f-K)} \end{bmatrix} \end{aligned} \quad (2.16)$$

with

$$-m_f m_A I_{K \times K} = A {}^t \tilde{A}, \quad AA^\dagger = \tilde{A}^\dagger \tilde{A} + \tilde{B}^* {}^t \tilde{B}.$$

The gauge group is Higgsed down to $U(N_c - K)$ where

$$K_{m_f \neq 0} \leq \min(N_c, N_f).$$

To understand the range of $K_{m_f \neq 0}$, first note that the Φ breaks the gauge group as $U(N_c) \rightarrow U(K) \times U(N_c - K)$. Now the $U(K)$ factor has effectively N_f massless flavors and because $\langle \tilde{Q}Q \rangle \neq 0$, $U(K)$ is further Higgsed down to $U(0)$.

For $m_f = 0$, we have $\Phi = 0$ and Q, \tilde{Q} are still of the above form (2.16) with one special requirement: $\tilde{A} = 0$. Because of this we have

$$K_{m_f=0} \leq \min\left(N_c, \left\lfloor \frac{N_f}{2} \right\rfloor\right),$$

where $\lfloor \cdot \rfloor$ means the integer part. The integer $K_{m_f=0}$ precisely corresponds to the r -th branch discussed in [10, 13]. The $m_f = 0$ case is different from the $m_f \neq 0$ case as follows. First,

Φ does not break the $U(N_c)$ gauge group, i.e., $U(N_c) \rightarrow U(N_c)$. Secondly, the r -th branch is the intersection of the Coulomb branch in which $\langle \tilde{Q}Q \rangle = 0$ and the Higgs branch in which $\langle \tilde{Q}Q \rangle \neq 0$, whereas for $m_f \neq 0$ the vacuum expectation value $\langle \tilde{Q}Q \rangle$ must be nonzero and the gauge group must be Higgsed down. For these reasons, $K_{m_f \neq 0}$ and $K_{m_f = 0}$ have different ranges.

The above classical classification of r -th branches is also valid in the quantum theory (including the baryonic branch).

The quantum r -th branch can also be discussed by using the Seiberg–Witten curve. In the r -th branch, the curve factorizes as

$$\begin{aligned} y_{N=2}^2 &= P_{N_c}(x)^2 - 4\Lambda^{2N_c-N_f}(x+m_f)^{N_f} \\ &= (x+m_f)^{2r} \left[P_{N_c-r}^2(x) - 4\Lambda^{2N_c-N_f}(x+m_f)^{N_f-2r} \right]. \end{aligned}$$

Because $N_c - r \geq 0$ (coming from $P_{N_c-r}(x)$) and $N_f - 2r \geq 0$ (coming from the last term), we have $r \leq N_c$ and $r \leq N_f/2$, which leads to the range

$$r \leq \min \left(N_c, \left\lfloor \frac{N_f}{2} \right\rfloor \right). \quad (2.17)$$

The relation between this classification of the Seiberg–Witten curve and the above classification of r -branches, in the $m_f \neq 0$ and $m_f = 0$ cases, is as follows. In the $m_f = 0$ case, we have one-to-one correspondence where the r is identified with $K_{m_f=0}$. In the $m_f \neq 0$ case, for a given r of the curve, there exist two cases: either $K_{m_f \neq 0} = r$ for $K_{m_f \neq 0} \leq [N_f/2]$, or $K_{m_f \neq 0} = N_f - r$ for $K_{m_f \neq 0} \geq [N_f/2]$.

2.3 The work of IKRSV

Now we discuss another aspect of the model. In the above, we saw that there is a period condition (2.11) for $T(z)$. So, if for the i -th cut we have $\oint_{A_i} T(z)dz = N_{c,i} = 0$, then it seems that, in the string theory realization of the gauge theory, there is no RR flux provided by D5-branes through this cut and the cut is closed. Because of this, it seems that we should set the corresponding glueball field $S_i = 0$. Based on this naive expectation, Ref. [4] calculated the effective superpotential of $USp(2N_c)$ theory with an antisymmetric tensor by the matrix model, which turned out to be different from the known results obtained by holomorphy and symmetry arguments (later Refs. [17, 18] confirmed this discrepancy).

This puzzle intrigued several papers [19, 20, 3, 15, 21, 22, 23, 24]. In particular, in [20], it was found that although $N_{c,i} = 0$, we cannot set $S_i = 0$. The reason became clear by later studies. Whether a cut closes or not is related to the total RR flux which comes from

both D5-branes and orientifolds. For $USp(2N_c)$ theory with antisymmetric tensor, although the RR flux from D5-branes is zero, there exists RR flux coming from the orientifold with positive RR charges, thus the cut does not close. That the cut does not close can also be observed from the Seiberg–Witten curve [15] where for such a cut, we have two single roots in the curve, instead of a double root. All these results were integrated in [3] for $\mathcal{N} = 1$ gauge theory with adjoint. Let us define $\widehat{N}_c = N_c$ for $U(N_c)$, $N_c/2 - 1$ for $SO(N_c)$ and $2N_c + 2$ for $USp(2N_c)$. Then the conclusion of [3] can be stated as

If $\widehat{N}_c > 0$, we should include S_i and extremize $W_{\text{eff}}(S_i)$ with respect to it. On the other hand, if $\widehat{N}_c \leq 0$, we just set $S_i = 0$ instead.

In [3], it was argued that this prescription of setting $S_i = 0$ can be explained in string theory realization by considering an extra degree of freedom which corresponds to the D3-brane wrapping the blown up S^3 and becomes massless in the $S \rightarrow 0$ limit [25]. Our second motivation of this paper is to generalize this conclusion to the case with flavors. We will discuss the precise condition when one should set $S_i = 0$ in order to get agreement with the gauge theory result, in the case with flavors.

2.4 Prospects from the strong coupling analysis

Before delving into detailed calculations, let us try to get some general pictures from the viewpoint of factorization of the Seiberg–Witten curve. Since we hope to generalize IKRSV, we are interested in the case where some S_i vanish. Because S_i is related to the size of a cut in the matrix model curve, which is essentially the same as the Seiberg–Witten curve, we want some cuts to be closed in the Seiberg–Witten curve. Namely, we want a double root in the factorization of the curve, instead of two single roots.

For $U(N_c)$ theory with N_f flavors of the same mass $m_f = -z_f$, tree level superpotential (2.1), and breaking pattern $U(N_c) \rightarrow \prod_{i=1}^n U(N_{c,i})$, the factorization form of the curve is [13]

$$\begin{aligned} P_{N_c}(z)^2 - 4\Lambda^{2N_c - N_f}(z - z_f)^{N_f} &= H_{N-n}(z)^2 F_{2n}(z), \\ F_{2n}(z) &= W'(z)^2 + f_{n-1}(z), \end{aligned} \tag{2.18}$$

where the degree $2n$ polynomial $F_{2n}(z) = W'(z)^2 + f_{n-1}(z)$ generically has $2n$ single roots. How can we have a double root instead of two single roots?

For a given fixed mass, for example $z_f = a_1$, there are three cases where we have a double root, as follows. (a) There is no $U(N_{c,1})$ group factor associated with the root a_1 , namely $N_{c,1} = 0$. (b) The $U(N_{c,1})$ factor is in the baryonic branch. This can happen for

$N_{c,1} \leq N_f < 2N_{c,1}$. (c) The $U(N_{c,1})$ factor is in the r -th non-baryonic branch with $r = N_{c,1}$. This can happen only for $N_f \geq 2N_{c,1}$. Among these three cases, (b) and (c) [13] are new for theories with flavors, and will be the focus of this paper. However, it is worth pointing out that the factorization form in the cases (b) and (c) are not the one given in (2.18) for a fixed mass, but the one given in (A.3).

Can we keep the factorization form (2.18) while having an extra double root? We can, but instead of a fixed mass we must let the mass “floating,” which means the following. There will be multiple solutions to the factorization form (2.18), and for any given solution the $2n$ single roots of $F_{2n}(x)$, denoted by a_i^\pm , $i = 1, \dots, n$, are functions of z_f . Now, we tune z_f so that $a_i^+(z_f) = a_i^-(z_f)$, i.e., so that two single roots combine into one double root. Since for different solutions this procedure will lead to different values of z_f , we call this situation the “floating” mass.

Now we have two ways to obtain extra double roots: one is with a fixed mass, but to go to the baryonic or the $r = N_{c,1}$ non-baryonic branch, while the other is to start with a general non-baryonic branch but using a floating mass. In fact it can be shown that these two methods are equivalent to each other when the double root is produced. In the calculations in section 4 we will use the floating mass to check our proposal.

3 One cut model—quadratic tree level superpotential

In this section we will study whether a cut closes up if one tries to pass too many poles through it. If the poles are near the cut, the precise form of the tree level superpotential (namely the polynomials $W(z)$, $m^{\tilde{f}}_f(z)$) is inessential and we can simplify the problem to the quadratic tree level superpotential given by (2.14). For this superpotential, we will compute the effective glueball superpotential using the formalism reviewed in the previous section. Then, by solving the equation of motion, we study the *on-shell* process of sending poles through the cut, and see whether the poles can pass or not.

Also, on the way, we make an observation on the relation between the exact superpotential and the vacuum expectation value of the tree level superpotential.

3.1 The off-shell W_{eff} , $M(z)$ and $T(z)$

First, let us compute the effective glueball superpotential for the quadratic superpotential (2.14). The matrix model curve (2.7) is related to $W'(z)$ in this case as

$$y_m^2 = W'(z)^2 + f_0(z) = m_A^2 z^2 - \mu \equiv m_A^2 (z^2 - \tilde{\mu}), \quad \tilde{\mu} = \frac{\mu}{m_A^2}. \quad (3.1)$$

Let us consider the case with K poles on the first sheet at $q_I = q_f$, for which $r_I = 1$, and with $(N_f - K)$ poles on the second sheet at $\tilde{q}_I = \tilde{q}_f$, for which $r_I = 0$ (recall that (q_f, \tilde{q}_f) is the lift of z_f defined in (2.15)). Using the curve (3.1) and various formulas summarized in the previous section, one can compute

$$\begin{aligned}
S &= \frac{1}{2\pi i} \oint_A R(z) dz = \frac{m_A \tilde{\mu}}{4} = \frac{\mu}{4m_A}, \\
\Pi &= 2 \int_{\sqrt{\tilde{\mu}}}^{\Lambda_0} y(z) dz = m_A \Lambda_0^2 - 2S - 2S \log \frac{\Lambda_0^2 m_A}{S}, \\
\Pi_{f,I}^{r_I=0} &= \int_{\tilde{q}_I}^{\tilde{\Lambda}_0} y(z) dz = - \int_{q_I}^{\Lambda_0} y(z) dz \\
&= \frac{-m_A \Lambda_0^2}{2} - 2S \log \frac{z_I}{\Lambda_0} + 2S \left[-\log \left(\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4S}{m_A z_I^2}} \right) + \frac{m_A z_I^2}{4S} \sqrt{1 - \frac{4S}{m_A z_I^2}} + \frac{1}{2} \right], \\
\Pi_{f,I}^{r_I=1} &= \int_{q_I}^{\tilde{\Lambda}_0} y(z) dz = - \int_{\tilde{q}_I}^{\Lambda_0} y(z) dz \\
&= \frac{-m_A \Lambda_0^2}{2} - 2S \log \frac{z_I}{\Lambda_0} + 2S \left[-\log \left(\frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{4S}{m_A z_I^2}} \right) - \frac{m_A z_I^2}{4S} \sqrt{1 - \frac{4S}{m_A z_I^2}} + \frac{1}{2} \right],
\end{aligned}$$

where we dropped $\mathcal{O}(1/\Lambda_0)$ terms. We have traded q_I, \tilde{q}_I for z_I in the square roots, so that the sign convention is such that $\sqrt{1 - \frac{4S}{m_A z_I^2}} \sim 1 - \frac{2S}{m_A z_I^2}$ and $\sqrt{z_I^2 - \frac{4S}{m_A}} \sim z_I$ for $|z_I|$ very large. Substituting this into (2.12), we obtain

$$\begin{aligned}
W_{\text{eff}}(S) &= S \left[N_c + \log \left(\frac{m_A^{N_c} \Lambda^{2N_c - N_f} \prod_I z_I}{S^{N_c}} \right) \right] \\
&\quad - \sum_{I, r_I=0} S \left[-\log \left(\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4S}{m_A z_I^2}} \right) + \frac{m_A z_I^2}{4S} \left(\sqrt{1 - \frac{4S}{m_A z_I^2}} - 1 \right) + \frac{1}{2} \right] \\
&\quad - \sum_{I, r_I=1} S \left[-\log \left(\frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{4S}{m_A z_I^2}} \right) + \frac{m_A z_I^2}{4S} \left(-\sqrt{1 - \frac{4S}{m_A z_I^2}} - 1 \right) + \frac{1}{2} \right], \quad (3.2)
\end{aligned}$$

where Λ is the dynamical scale of the corresponding $\mathcal{N} = 2$ gauge theory defined in (2.13).

Let us compute resolvents also. The resolvent $M(z)$ is an $N_f \times N_f$ matrix. Using (2.8), we find that for the I -th eigenvalue

$$M_I(z) = \frac{-R(z)}{(z - q_I)} + \frac{(1 - r_I)R(q_I)}{(z - q_I)} + \frac{r_I R(\tilde{q}_I)}{(z - q_I)}.$$

Expanding $M_I(z)$ around $z = \infty$ we can read off the following vacuum expectation values

$$\begin{aligned}\langle \tilde{Q}Q \rangle_I &= \frac{m_A}{2} \left[z_I + (2r_I - 1) \sqrt{z_I^2 - \tilde{\mu}} \right], \quad \langle \tilde{Q}\Phi Q \rangle_I = \frac{m_A}{4} \left[2z_I^2 + 2(2r_I - 1)z_I \sqrt{z_I^2 - \tilde{\mu}} - \tilde{\mu} \right], \\ \left\langle \sum_I -(\tilde{Q}\Phi Q - z_I \tilde{Q}Q) \right\rangle &= \frac{N_f m_A \tilde{\mu}}{4} = N_f S.\end{aligned}$$

There is something worth noting here. One might naively expect that the exact superpotential is simply the vacuum expectation value of the tree level superpotential (2.14), as is the case without flavors. However, this naive expectation is wrong! Although we have $\langle W_{\text{tree,fund}} \rangle = \langle -\sum_I (\tilde{Q}\Phi Q - z_I \tilde{Q}Q) \rangle \neq 0$ if $S \neq 0$, we still have

$$W_{\text{eff,on-shell}} = \left\langle \frac{m_A}{2} \text{Tr } \Phi^2 \right\rangle, \quad (3.3)$$

as we will see shortly. The reason for (3.3) can be explained by symmetry arguments [26, 27]. Although the tree level part $\langle W_{\text{tree,fund}} \rangle$ for fundamentals is generically nonzero and also contributes to W_{low} , the contribution is precisely canceled by the dynamically generated superpotential W_{dyn} , leaving only the Φ part of W_{tree} ⁴.

Let us calculate the resolvent $T(z)$ also. For the present example, $c(z) = m_A(N_c - \frac{N_f}{2})$ from (2.10). Therefore, using (2.9), we obtain the expansion of the resolvent $T(z)$:

$$\begin{aligned}T(z) &= \frac{N_c}{z} + \sum_I \left[\frac{z_I}{2} + \frac{2r_I - 1}{2} \sqrt{z_I^2 - \tilde{\mu}} \right] \frac{1}{z^2} \\ &\quad + \left[\frac{\tilde{\mu}(2N_c - N_f)}{4} + \sum_I \left(\frac{z_I^2}{2} + \frac{2r_I - 1}{2} z_I \sqrt{z_I^2 - \tilde{\mu}} \right) \right] \frac{1}{z^3} + \dots\end{aligned} \quad (3.4)$$

From this we can read off $\langle \text{Tr } \Phi^n \rangle$. For example, for $K = 0$ we have

$$\langle \text{Tr } \Phi \rangle = \frac{N_f}{2} \left(z_f - \sqrt{z_f^2 - \tilde{\mu}} \right), \quad \langle \text{Tr } \Phi^2 \rangle = \frac{N_f}{2} z_f \left(z_f - \sqrt{z_f^2 - \tilde{\mu}} \right) + \frac{(2N_c - N_f)\tilde{\mu}}{4}.$$

3.2 The on-shell solution

Now we can use the above off-shell expressions to find the on-shell solution. First we rewrite the superpotential as

$$\begin{aligned}W_{\text{eff}} &= S \left[N_c + \log \left(\frac{\Lambda_1^{3N_c - N_f}}{S^{N_c}} \right) \right] \\ &\quad - (N_f - K)S \left[-\log \left(\frac{z_f}{2} + \frac{1}{2} \sqrt{z_f^2 - \frac{4S}{m_A}} \right) + \frac{m_A z_f}{4S} \left(\sqrt{z_f^2 - \frac{4S}{m_A}} - z_f \right) + \frac{1}{2} \right] \\ &\quad - KS \left[-\log \left(\frac{z_f}{2} - \frac{1}{2} \sqrt{z_f^2 - \frac{4S}{m_A}} \right) + \frac{m_A z_f}{4S} \left(-\sqrt{z_f^2 - \frac{4S}{m_A}} - z_f \right) + \frac{1}{2} \right],\end{aligned} \quad (3.5)$$

⁴We would like to thank K. Intriligator for explaining this point to us.

where $\Lambda_1^{3N_c-N_f} \equiv m_A^{N_c} \Lambda^{2N_c-N_f}$. We have set all z_I to be z_f , i.e., all masses are the same, as in (2.15). From this we obtain ⁵

$$0 = \log\left(\frac{\Lambda_1^{3N_c-N_f}}{S^{N_c}}\right) + K \log\left(\frac{z_f - \sqrt{z_f^2 - \frac{4S}{m_A}}}{2}\right) + (N_f - K) \log\left(\frac{z_f + \sqrt{z_f^2 - \frac{4S}{m_A}}}{2}\right), \quad (3.6)$$

or ⁶

$$0 = \log\left(\widehat{S}^{-N_c}\right) + K \log\left(\frac{\widehat{z}_f - \sqrt{\widehat{z}_f^2 - 4\widehat{S}}}{2}\right) + (N_f - K) \log\left(\frac{\widehat{z}_f + \sqrt{\widehat{z}_f^2 - 4\widehat{S}}}{2}\right), \quad (3.7)$$

where we have defined dimensionless quantities $\widehat{S} = \frac{S}{m_A \Lambda^2} = \frac{\tilde{\mu}}{4\Lambda^2}$ and $\widehat{z}_f = \frac{z_f}{\Lambda}$. Note that using these massless quantities the cut is from along the interval $[-2\sqrt{\widehat{S}}, 2\sqrt{\widehat{S}}]$.

Using (3.6) or (3.7) it is easy to show that

$$W_{\text{eff, on-shell}} = \left(N_c - \frac{N_f}{2}\right) S + \frac{m_A z_f^2}{4} \left[N_f + (2K - N_f) \sqrt{1 - \frac{4S}{m_A z_f^2}} \right].$$

Also by expanding $T(z)$ in the present case, just as we did in (3.4), we can read off

$$\left\langle \frac{m_A}{2} \text{Tr} \Phi^2 \right\rangle = \left(N_c - \frac{N_f}{2}\right) S + \frac{m_A z_f^2}{4} \left[N_f + (2K - N_f) \sqrt{1 - \frac{4S}{m_A z_f^2}} \right],$$

which gives us the relation

$$\left\langle \frac{m_A}{2} \text{Tr} \Phi^2 \right\rangle = W_{\text{eff, on-shell}}$$

as we promised in (3.3).

Equation (3.6) is hard to solve. But if we want just to discuss whether the cut closes up when we bring $z_f \rightarrow 0$, we can set $K = 0$ ⁷, for which (3.6) reduces to

$$z_f = \omega_{N_f}^r \widehat{S}^{\frac{N_c}{N_f}} + \omega_{N_f}^{-r} \widehat{S}^{\frac{N_f - N_c}{N_f}}. \quad (3.8)$$

⁵It is easy to check that the equation (3.6) with parameters (N_c, N_f, K) is the same as the one with parameters $(N_c - r, N_f - 2r, K - r)$. Also from the expression (3.4) it is straightforward to see we have $\langle \text{Tr} \Phi^2 \rangle_{N_c, N_f, K} = \langle \text{Tr} \Phi^2 \rangle_{N_c - r, N_f - 2r, K - r} + r z_f^2$. All of these facts are the result of the “addition map” observed in [13]. Furthermore, one can show that both (3.6) and (3.4) for $K = 0$ are exactly the same as the one given by the strong coupling analysis in [16] and the weak coupling analysis in [28, 29].

⁶As we mentioned before, for $m_f \neq 0$ the allowed Higgs branch requires $K \leq N_f$ but the strong coupling analysis gives $K \leq N_f/2$. The resolution for that puzzle is that if $K > N_f/2$, it is given by $(N_f - K)$ -th branch of the curve. Since the same r -th branch of the curve gives both r and $(N_f - r)$ Higgs branches, we expect that r and $(N_f - r)$ Higgs branches are related. This relation is given by $\tilde{S} = S b^2$, $\tilde{z}_f = z_f b$ with $b = \frac{m_A \Lambda^2}{S}$. It can be shown that with the above relation, the equation of motion of S for the K -th branch is changed to the equation of motion of \tilde{S} for the $(N_f - K)$ -th branch.

⁷If $K \neq 0$, we will have $U(N_c) \rightarrow U(K) \times U(N_c - K)$ and the problem reduces to of $U(N_c - K)$ with $(N_f - 2K)$ flavors in the 0-th branch.

Here, ω_{N_f} is the N_f -th root of unity, $\omega_{N_f} = e^{2\pi i/N_f}$, and $r = 0, 1, \dots, (N_f - 1)$ corresponds to different branches of solutions. It is also amusing to note that above solution has the Seiberg duality [30] where electric theory with (N_c, N_f) is mapped to a magnetic theory with $(N_f - N_c, N_f)$.

With these preparations, we can start to discuss the on-shell process of passing poles through the cut from the second sheet.

3.3 Passing poles through a branch cut

Consider moving N_f poles on top of each other at infinity on the second sheet toward the cut along a line passing through the origin and making an angle of θ with the real axis. Namely, take⁸

$$z(\tilde{q}_f) = z(q_f) = pe^{i\theta}, \quad p, \theta \in \mathbb{R}, \quad (3.9)$$

and change p from $p = \infty$ to $p = -\infty$ (see Fig. 1). This equation (3.9) needs some explanation. Remember that $z_f = z(q_f) = z(\tilde{q}_f)$ denotes the projection from the Riemann surface Σ (Eq. (3.1)) to the z -plane. For each point z on the z -plane, there are two corresponding points: q on the first sheet and \tilde{q} on the second sheet. Although we are starting with poles at \tilde{q}_f on the second sheet, we do not know in advance if the poles will pass through the cut and end up on the first sheet, or it will remain on the second sheet. Therefore we cannot specify which sheet the poles are on, and that is why we used $z(q_f)$, $z(\tilde{q}_f)$ in (3.9), instead of q_f or \tilde{q}_f .

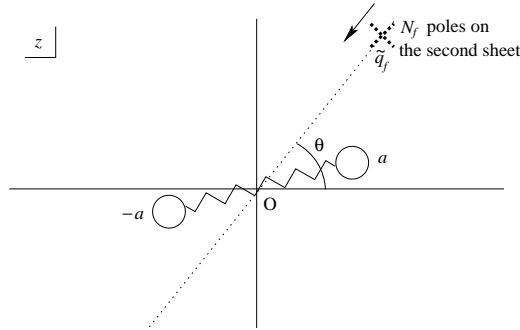


Figure 1: A process in which N_f poles at \tilde{q}_f on the second sheet far away from the cut approach the branch cut on the double sheeted z -plane, along a line which goes through the origin and makes an angle θ with the real z axis. The “ \times ” with dotted lines denotes the poles on the second sheet, moving in the direction of the arrow. The two branch points $\pm a$ are connected by the branch cut, which is denoted by a zigzag.

⁸Throughout this subsection, we will use the dimensionless quantities \hat{z}_f , \hat{S} , etc. and omit the hats on them to avoid clutter, unless otherwise mentioned.

Below, we study the solution to the equation of motion (3.8), changing $p \in \mathbb{R}$ from $p = \infty$ to $p = -\infty$. By redefining z_f, S by $z_f \rightarrow z_f e^{2\pi i r / (N_f - 2N_c)}, S \rightarrow S e^{4\pi i r / (N_f - 2N_c)}$, we can bring (3.8) to the following form:

$$z_f = p e^{i\theta} = S^t + S^{1-t}, \quad (3.10)$$

where

$$\frac{N_c}{N_f} \equiv t.$$

Henceforth we will use (3.10). Because z_f as well as S is complex, the position of the branch points (namely, the ends of the cut), $\pm a$, where

$$a \equiv \sqrt{4S}$$

is also complex, which means that in general the cut makes some finite angle with the real axis, as shown in Fig. 1.

• $N_f = N_c$

As the simplest example, let us first consider the $N_f = N_c$ (i.e., $t = 1$) case. We will see that the poles barely pass through the cut but get soon bounced back to the second sheet.

The equation of motion (3.10) is, in this case,

$$z_f = p e^{i\theta} = S + 1. \quad (3.11)$$

Therefore, as we change p , the position of the branch points changes according to

$$a = \sqrt{4S} = 2\sqrt{z_f - 1} = 2\sqrt{p e^{i\theta} - 1}. \quad (3.12)$$

Let us look closely at the process, step by step. The point is that transition between the first and the second sheet can happen only when the cut becomes parallel to the incident direction of the poles, or when the poles pass through the origin.

(1) $p \simeq +\infty$, on the second sheet:

In this case, we can approximate the right hand side of (3.12) as

$$a = 2p^{\frac{1}{2}} e^{\frac{i\theta}{2}} (1 - p^{-1} e^{-i\theta})^{\frac{1}{2}} \simeq 2p^{\frac{1}{2}} e^{\frac{i\theta}{2}} e^{-\frac{1}{2} p^{-1} e^{-i\theta}} = 2p^{\frac{1}{2}} e^{-\frac{1}{2} p^{-1} \cos \theta} e^{i(\frac{\theta}{2} + \frac{1}{2} p^{-1} \sin \theta)}.$$

Therefore, when the poles are far away, the angle between the cut and the real axis is approximately $\frac{\theta}{2} > 0$ (we assume $0 < \theta < \frac{\pi}{2}$). Furthermore, as the poles approach (p becomes smaller), the cut shrinks (because of $p^{\frac{1}{2}}$) and rotates counterclockwise (because of $e^{\frac{i}{2} p^{-1} \sin \theta}$). This corresponds to Fig. 2a.

- (2) Because the cut is rotating counterclockwise, as the poles approach, the cut will eventually become parallel to the incident direction, at some point. This happens when

$$a^2 = 4(pe^{i\theta} - 1) = 4[(p \cos \theta - 1) + ip \sin \theta] \propto e^{2i\theta} = \cos 2\theta + i \sin 2\theta.$$

By simple algebra, one obtains

$$p = 2 \cos \theta, \quad a = 2e^{i\theta}. \quad (3.13)$$

Note that this is the only solution; the cut becomes parallel to the incident direction only once. Because $0 < p < |a| = 2$ (we are assuming $0 < \theta < \pi/2$), by the time the cut becomes parallel to the incident direction, the poles have come inside of the interval $[-2e^{i\theta}, 2e^{i\theta}]$, along which the cut extends when it is parallel to the incident direction. This implies that the poles cross⁹ the cut at this point, and enter into the first sheet. Fig. 2b shows the situation when this transition is about to happen. In Fig. 2c, the poles are just crossing the cut. Fig. 2d corresponds to the situation just after the transition happened; the poles have passed the cut and are now proceeding on the first sheet.

Here we implicitly assumed that the cut is still rotating counterclockwise with a finite angular velocity, but this can be shown by expanding a around (3.13) as $p = 2 \cos \theta + \Delta p$. A short computation shows

$$a \simeq 2e^{\frac{1}{2}\Delta p \cos \theta} e^{i\theta - \frac{i}{2}\Delta p \sin \theta}$$

which implies that the cut shrinks and rotates counterclockwise if we move the poles to the left (Δp decreases).

- (3) $p \simeq 0$:

If the poles proceed on the real line further, it eventually reaches the origin $p = 0$. By expanding (3.12) around $p = 0$, one obtains

$$a = 2(e^{\pi i} + pe^{i\theta})^{\frac{1}{2}} \simeq 2e^{-\frac{1}{2}p \cos \theta} e^{i(\frac{\pi}{2} - \frac{1}{2}p \sin \theta)}. \quad (3.14)$$

Therefore the cut has a finite size ($|a| = 2$) at $p = 0$ and along the imaginary axis, still rotating counterclockwise, but now expanding. Because the cut goes through the origin, the poles pass through the cut again and comes back onto the second sheet (Fig. 2e).

- (4) $p \simeq -\infty$, on the second sheet:

If the poles have gone far past the cut so that $p < 0$, $|p| \gg 1$, we can approximate

⁹As mentioned in footnote 2, there is no real physical meaning to the position of the cut itself.

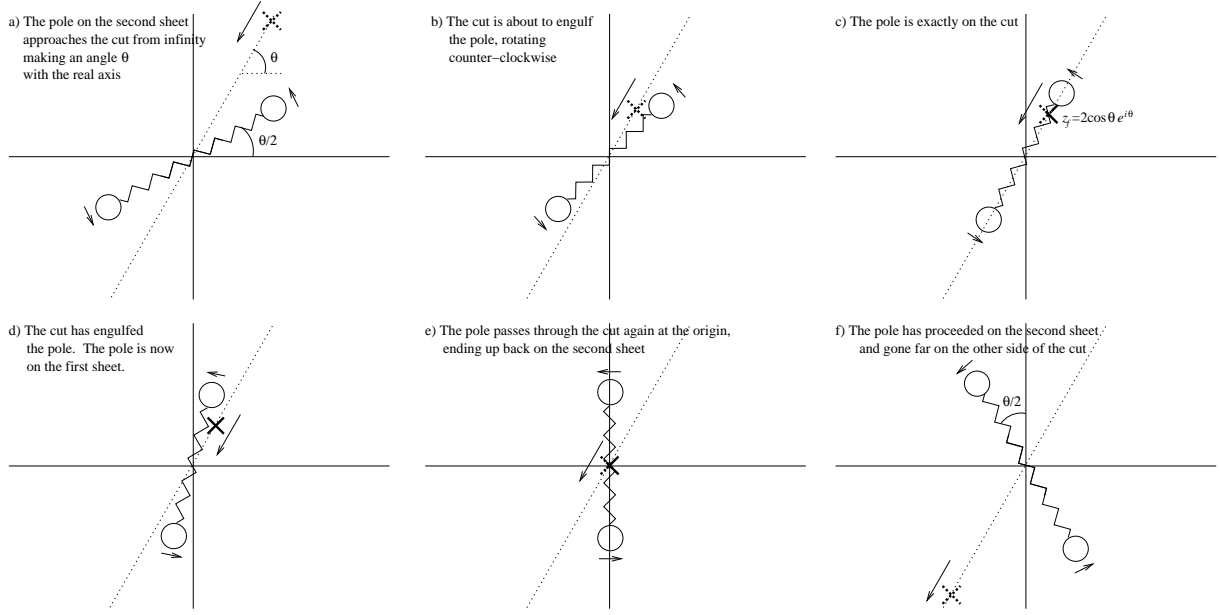


Figure 2: Six configurations of the branch cut and the poles. The poles are depicted by “ \times ” and moving along a line at an angle θ with the real axis, as the arrow on it indicates. The “ \times ” in solid (dotted) lines denotes poles on the first (second) sheet. The branch cut is rotating counterclockwise (as the arrows on its sides indicate), changing its length.

(3.12), as before, as

$$a = 2|p|^{\frac{1}{2}} e^{\frac{i}{2}(\theta+\pi)} (1 - p^{-1} e^{-i\theta})^{\frac{1}{2}} \simeq 2|p|^{\frac{1}{2}} e^{\frac{i}{2}} |p|^{-1} \cos \theta e^{i(\frac{\pi}{2} + \frac{\theta}{2} - \frac{1}{2}|p|^{-1} \sin \theta)}.$$

Therefore, as the poles go away, the cut expands and rotates counterclockwise. The angle between the cut and the real axis asymptotes to $(\frac{\pi}{2} + \frac{\theta}{2})$ (Fig. 2f).

In the above we assumed that $0 < \theta < \frac{\pi}{2}$. If $\frac{\pi}{2} < \theta < \pi$, the only difference is that the order of steps (2) and (3) are exchanged. If $\theta < 0$, the cut rotates clockwise instead of counterclockwise.

When is the cut shortest in this whole process? From (3.12), one easily obtains

$$|a| = 2[(p - \cos \theta)^2 + \sin^2 \theta]^{1/4} \geq 2|\sin \theta|^{1/2}. \quad (3.15)$$

Therefore, when the poles are at $z_f = p e^{i\theta} = \cos \theta e^{i\theta}$, which is between the steps (2) and (3) above, the cut becomes shortest. In particular, in the limit $\theta \rightarrow 0$ or $\theta \rightarrow \pm\pi$, the cut completely closes up instantaneously. These correspond to configurations with either a horizontal cut with poles colliding sideways, or a vertical cut with poles colliding from right

above or from right below. Actually the existence of the $S = 0$ solution is easy to see in (3.11): it is just $z_f = 1, S = 0$.

Summary: for $N_f = N_c$, when one moves poles on the second sheet from infinity along a line toward a cut, poles pass through the cut onto the first sheet and move away from the cut by a short distance. Then poles are bounced back to the second sheet again. Therefore, one can never move poles far away from the cut on the first sheet. During the process, in certain situations, the cut completely closes up.

• $N_f \neq N_c$

Now let us consider a more general case with $N_f \neq N_c$. We again consider a situation where poles on the second sheet approach a cut. This time we will be brief and sketchy, because a detailed analysis such as the one we did for the $N_f = N_c$ case would be rather lengthy due to the existence of multiple branches, and would not be very illuminating.

First, let us ask how we can see whether poles are on the first sheet or on the second sheet, from the behavior of S versus p . Because this is not apparent in the equation of motion of the form (3.10), let us go back to

$$0 = \frac{\partial W_{\text{eff}}}{\partial S} \propto \log S^{-N_c} + N_f \log \frac{z_f \mp \sqrt{z_f^2 - 4S}}{2} \implies S^t = \frac{z_f \mp \sqrt{z_f^2 - 4S}}{2}. \quad (3.16)$$

which led to the equation (3.10). Here the “−” (“+”) sign corresponds to q_f on the first (second) sheet. For $|z_f|^2 \gg |4S|$, the square root can be approximated as $\sqrt{z_f^2 - 4S} = z_f(1 - 4S/z_f^2)^{1/2} \simeq z_f(1 - 2S/z_f^2)$ (our sign convention was discussed above (3.2)). Therefore (3.16) is, on the first sheet,

$$S^t \simeq \frac{z_f - z_f(1 - 2S/z_f^2)}{2} = \frac{S}{z_f} \implies z_f \simeq S^{1-t}, \quad (3.17)$$

while on the second sheet

$$S^t \simeq \frac{z_f + z_f(1 - 2S/z_f^2)}{2} \simeq z_f \implies z_f \simeq S^t. \quad (3.18)$$

Now let us solve (3.10) for $|z_f| \gg 1$. By carefully comparing the magnitude of the two terms in (3.10), one obtains

$$\begin{aligned} N_f < N_c \quad (1 < t) &\rightarrow \begin{cases} z_f \simeq S^t & \rightarrow |S| \simeq |p|^{\frac{1}{t}}, & |S| \gg 1 \\ z_f \simeq S^{1-t} & \rightarrow |S| \simeq |p|^{-\frac{1}{t-1}}, & |S| \ll 1 \end{cases} \\ N_c < N_f < 2N_c \quad (\frac{1}{2} < t < 1) &\rightarrow z_f \simeq S^t \rightarrow |S| \simeq |p|^{\frac{1}{t}}, & |S| \gg 1 \\ 2N_c < N_f \quad (0 < t < \frac{1}{2}) &\rightarrow z_f \simeq S^{1-t} \rightarrow |S| \simeq |p|^{\frac{1}{1-t}}, & |S| \gg 1 \end{aligned} \quad (3.19)$$

It is easy to show that $|z_f|^2 \gg |4S|$ in all cases. So by using (3.17), (3.18), we conclude that the first and the third lines in (3.19) correspond to poles on the second sheet, while the second and the last lines correspond to poles on the first sheet. This implies that, only for $N_f < N_c$, poles on the second sheet can pass through the cut all the way and go infinitely far away on the first sheet from a cut, as we will see explicitly in the examples below. For $N_c < N_f < 2N_c$, if one tries to pass poles through a cut, then either poles will be bounced back to the second sheet, or the cut closes up before the poles reach it. For $N_f > 2N_c$, there is no solution corresponding to poles moving toward a cut from infinity on the second sheet. This should be related to the fact that in this case glueball S is not a good IR field. Therefore, this one cut model is not applicable for $N_f > 2N_c$.

To argue that these statements are true, rather than doing an analysis similar to the one we did for the $N_f = N_c$ case, we will present some explicit solutions for some specific values of $t = \frac{N_c}{N_f}$ and θ , and argue general features. Before looking at explicit solutions, note that there are multiple solutions to Eq. (3.10) which can be written as

$$z_f = (S^{\frac{1}{N_f}})^{N_c} + (S^{\frac{1}{N_f}})^{N_f - N_c}. \quad (3.20)$$

From the degree of this equation, one sees that the number of the solutions to (3.20) is:

$$\begin{aligned} N_f &\leq N_c &\implies & 2N_c - N_f \text{ solutions,} \\ N_c &\leq N_f \leq 2N_c &\implies & N_c \text{ solutions,} \\ 2N_c &\leq N_f &\implies & N_f - N_c \text{ solutions.} \end{aligned}$$

Therefore, if we solve the equation of motion (3.10), in general we expect multiple branches of solutions. We do not have to consider the N_f branches of the root $S^{\frac{1}{N_f}}$, because it is taken care of by the phase rotation we did above (3.10).

Now let us look at explicit solutions for $U(2)$ example. For $N_f = \frac{1}{2}N_c$ ($t = 2$), the solution to the equation of motion (3.10) is

$$S = -\frac{1}{3} \left(\frac{27 + \sqrt{729 - 108z_f^3}}{2} \right)^{1/3} - \left(\frac{27 + \sqrt{729 - 108z_f^3}}{2} \right)^{-1/3},$$

where three branches of the cubic root are implied. In Fig. 3 we plotted $|S|$ versus p for these branches, for a randomly chosen value of the angle of incidence, $\theta = \pi/6$. Even if one changes θ , there are always three branches whose general shapes are similar to the ones in Fig. 3. These three branches changes into one another when θ is changed by $2\pi/3$. One can easily see which branch corresponds to what kind of processes, by the fact that on the first sheet $|S| \ll 1$ as $|p| \rightarrow \infty$, while on the second sheet $|S| \gg 1$ as $|p| \rightarrow \infty$. The three branches

correspond to the process in which: i) poles go from the second sheet to the first sheet through the cut, without any obstruction, ii) poles go from the first sheet to the second sheet (this is not the process we are interested in), and iii) poles coming from the second sheet get reflected back to the second sheet. Note that the cut has never closed in all cases, because $|S|$ is always nonvanishing.

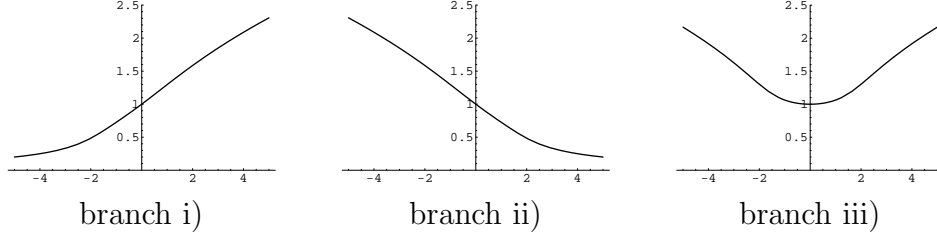


Figure 3: The graph of $|S|$ versus p for $N_f = \frac{1}{2}N_c$ ($t = 2$), $\theta = \pi/6$. The vertical axis is $|S|$ and the horizontal axis is p . Although we are showing just the $\theta = \pi/6$ case, there are similar looking three branches for any value of θ , which change into one another when θ is changed by $2\pi/3$.

Similarly, for $N_f = \frac{3}{2}N_c$ ($t = \frac{2}{3}$), the solution to the equation of motion (3.10) is

$$S = \frac{1}{2} [-3z_f - 1 \pm (z_f + 1)\sqrt{4z_f + 1}] .$$

This time there are two branches, which change into each other when the θ is changed by π . We plotted $|S|$ versus p for $\theta = \pi/2$ in Fig. 4. It shows two possibilities: i) poles coming from the second sheet get reflected back to the second sheet, for which $|S| \neq 0$ as $p \rightarrow 0$, ii) the cut closes up before poles passes through it, for which $|S| \rightarrow 0$ as $p \rightarrow 0$.

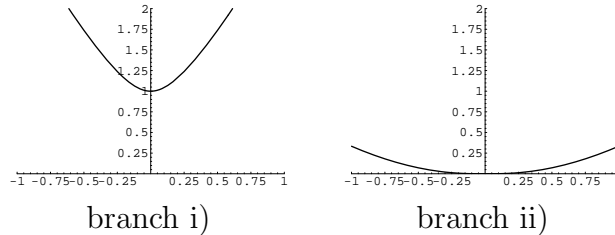


Figure 4: The graph of $|S|$ versus p for $N_f = \frac{3}{2}N_c$ ($t = \frac{2}{3}$) or $N_f = 3N_c$ ($t = \frac{1}{3}$), for $\theta = \pi/2$. The two values of t give the same graph. The vertical axis is $|S|$ and the horizontal axis is p . Although we are showing just the $\theta = \pi/2$ case, there are similar looking two branches for any value of θ , which change into each other when θ is changed by π ,

The $N_f = 3N_c$ ($t = \frac{1}{3}$) case is also described by the same Fig. 4. However, as we discussed below (3.19), it does not correspond to a process of poles approaching the cut from infinity on the second sheet; it corresponds to poles on the first sheet and we cannot give any physical interpretation to it.

These demonstrate the following general features:

- For $N_f < N_c$, one can move poles at infinity on the second sheet through a cut all the way to infinity on the first sheet without obstruction, if one chooses the incident angle appropriately. If the angle is not chosen appropriately, the poles will be bounced back to the second sheet.
- For $N_c \leq N_f < 2N_c$, one cannot move poles at infinity on the second sheet through a cut all the way to infinity on first sheet. If one tries to, either i) the cut rotates and sends the poles back to the second sheet, or ii) the cut closes up before the poles reach it.
- For $2N_c < N_f$, the one cut model does not apply directly.

$N_f = 2N_c$ is an exceptional case, for which the equation of motion (3.10) becomes

$$z_f = 2S^{1/2}. \quad (3.21)$$

Therefore $|S| \rightarrow 0$ as $z_f \rightarrow 0$, and the cut always closes before the poles reach it.

Subtlety in $S = 0$ solutions

If $p = 0$, or equivalently if $z_f = 0$, there is a subtle, but important point we overlooked in the above arguments. For $z_f = 0$, the superpotential (3.5) becomes

$$\begin{aligned} W_{\text{eff}} &= S \left[\left(N_c - \frac{N_f}{2} \right) + \log \left(\frac{(-1)^{N_f/2} m_A^{N_c - N_f/2} \Lambda^{2N_c - N_f}}{S^{N_c - N_f/2}} \right) \right] \\ &= S \left[\left(N_c - \frac{N_f}{2} \right) + \log \left(\frac{(-1)^{-N_f/2} m_A^{N_c - N_f/2} \Lambda_0^{2N_c - N_f}}{S^{N_c - N_f/2}} \right) \right] + 2\pi i \tau_0 S \\ &= \left(N_c - \frac{N_f}{2} \right) S \left[1 + \log \left(\frac{\tilde{\Lambda}_0^3}{S} \right) \right] + 2\pi i \tau_0 S \end{aligned} \quad (3.22)$$

with $\tilde{\Lambda}_0^{3(N_c - N_f/2)} \equiv (-1)^{-N_f/2} m_A^{N_c - N_f/2} \Lambda_0^{2N_c - N_f}$. Only from here to (3.23), S means the dimensionful quantity ($S = m_A \Lambda^2 \hat{S}$; see below (3.7)). In addition, in the second line of (3.22),

we rewrite the renormalized scale Λ in terms of the bare scale Λ_0 and the bare coupling τ_0 using the relation (2.13). In our case, $B_L = (-1)^{N_f}$. The equation of motion derived from (3.22) is

$$\left(N_c - \frac{N_f}{2}\right) \log \left(\frac{\tilde{\Lambda}_0^3}{S}\right) + 2\pi i \tau_0 = 0$$

and the solution is

$$S = \begin{cases} \tilde{\Lambda}_0^3 e^{\frac{2\pi i \tau_0}{N_c - N_f/2}} & N_f \neq 2N_c, \\ \text{no solution} & N_f = 2N_c. \end{cases} \quad (3.23)$$

For $N_f < N_c$, (3.23) is consistent with the fact that the $|S|$ versus p graphs in Fig. 3 all go through the point $(p, |S|) = (0, 1)$ (now S means the dimensionless quantity). Also for $N_f = N_c$, (3.23) is consistent with the result (3.14) ($|a| = 2$, so $|S| = 1$). On the other hand, for $N_f > N_c$, (3.23) implies that we should exclude the origin $(p, |S|) = (0, 0)$ from the $|S|-p$ graphs in Fig. 4, which is the only $S = 0$ solution (this includes the $N_f = 2N_c$ case (3.21)).¹⁰

Therefore, the above analysis seems to indicate that, for $N_f > N_c$, the $S = 0$ solution at $p = 0$, or equivalently $z_f = 0$ is an exceptional case and should be excluded. On the other hand, as can be checked easily, gauge theory analysis based on the factorization method shows that there is an $S = 0$ solution in the baryonic branch. Thus we face the problem of whether the baryonic $S = 0$ branch for $N_f > N_c$ can be described in matrix model, as alluded to in the previous discussions.

Note that, there is also an $S = 0$ solution for $N_c = N_f$ in certain situations, as discussed below (3.15). For this solution, which is in the non-baryonic branch, there is no subtlety in the equation of motion such as (3.23), and it appears to be a real on-shell solution. This will be discussed further below.

3.4 Generalization of IKRSV for the one cut model

In the above and in subsection 2.4, we argued that for $N_f \geq N_c$ the $S = 0$ solutions are real, on-shell solutions based on the factorization analysis. More accurately, there are two cases with $S = 0$: the one in the maximal non-baryonic branch with $N_f \geq 2N_c$ and the other one in the baryonic branch with $N_c \leq N_f < 2N_c$. The case of non-baryonic branch cannot be discussed in the one cut model, which is applicable only to $N_f < 2N_c$. On the other

¹⁰One may think that if one uses the first line of (3.22), then for $N_f = 2N_c$, $W_{\text{eff}} = S \log[(-1)^{N_f/2}]$ and there are solutions for some N_f . However, the glueball superpotential that string theory predicts [31, 32] is the third line of (3.22) which is in terms of the bare quantities Λ_0 and τ_0 . If $N_f = 2N_c$, then the log term vanishes and one cannot define a new scale Λ as we did in (2.13) to absorb the linear term $2\pi i \tau_0 S$.

hand, the baryonic one did show up in the previous subsection, but we just saw above that those solutions should be excluded by the matrix model analysis. What is happening? Is it impossible to describe the baryonic branch in matrix model?

Recall that the glueball field S has to do with the strongly coupled dynamics of $U(N_c)$ theory. That $S = 0$ in those solutions means that there is no strongly coupled dynamics any more, namely the $U(N_c)$ group has broken down completely. The only mechanism for that to happen is by condensation of a massless charged particle which makes the $U(1)$ photon of the $U(N_c)$ group massive. Therefore, in order to make $S = 0$ a solution, we should incorporate such an extra massless degree of freedom, which is clearly missing in the description of the system in terms only of the glueball S . This extra degree of freedom should exist even in the $N_f = N_c$ case where $S = 0$ really is an on-shell solution as discussed below (3.15); we just could not directly see the degree of freedom in this case.

The analysis of [3] hints on what this extra massless degree of freedom should be in the matrix model / string theory context. Note that, the superpotential (3.22) is of exactly the same form as Eq. (4.5) of [3], if we interpret $N_c - N_f/2 \equiv \widehat{N}$ as the amount of the net RR 3-form fluxes. In [3] it was argued that, if the net RR flux \widehat{N} vanishes, one should take into account an extra degree of freedom corresponding to D3-branes wrapping the blown up S^3 in the Calabi–Yau geometry [25], and condensation of this extra degree of freedom indeed makes $S = 0$ a solution to the equation of motion. The form of the superpotential (3.22) strongly suggests that the same mechanism is at work for $N_f = 2N_c$ in the $r = N_c$ non-baryonic branch; condensation of the D3-brane makes $S = 0$ a solution. Furthermore, as discussed in [3], for $N_f > 2N_c$ the glueball S is not a good variable and should be set to zero. A concise way of summarizing this conclusion is: if the generalized dual Coxeter number $h = N_c - N_f/2$ is zero or negative, we should set S to zero in the $r = N_c$ non-baryonic branch.

However, this is not the whole story, as we have discussed in subsection 2.4. As we saw above, we need some extra physics also for $N_c \leq N_f < 2N_c$ in order to explain the matrix model result in the baryonic branch. We argue below that this extra degree of freedom at least in the $N_c < N_f < 2N_c$ case should also be the D3-brane wrapping S^3 which shrinks to zero when the glueball goes to zero: $S \rightarrow 0$.

The original argument of [3] is not directly applicable for $N_f < 2N_c$ because there are nonzero RR fluxes penetrating such a D3-brane ($\widehat{N} \neq 0$). These RR fluxes induce fundamental string charge on the D3-brane. Because the D3-brane is compact, there is no place for the flux to end on (note that this flux is not the RR one but the one associated with the fundamental string charge). Hence it should emanate some number of fundamental strings. If there are no flavors, there is no place for such fundamental strings to end on, so they should extend

to infinity. This fact led to the conclusion of [3] that the D3-brane wrapping S^3 is infinitely massive and not relevant unless $\widehat{N} = 0$.

However, in our situation, there are places for the fundamental strings to end on — noncompact D5-branes which give rise to flavors [31, 32]. In particular, precisely in the $z_f = 0$ case, where we have $S = 0$ solutions for $N_c < N_f < 2N_c$, the D3-brane wrapping S^3 intersects the noncompact D5-branes in the $S \rightarrow 0$ limit, hence the 3-5 strings stretching between them are massless. Therefore the D3-brane with these fundamental strings on it is massless and should be included in the low energy description. It is well known [33] that such a D-brane with fundamental strings ending on it can be interpreted as baryons in gauge theory.¹¹ Condensation of this baryon degree of freedom should make $S = 0$ a solution, making the photon massive and breaking the $U(N_c)$ down to $U(0)$. The precise form of the superpotential for this extra degree of freedom must be more complicated than the one proposed in [3] for the case without flavors.

All these analyses tell us the following prescription:

Using the floating mass condition that all N_f poles are on top of one branch point¹² on the Riemann surface, we will have an $S = 0$ solution for $N_f \geq 2N_c$. For $N_c < N_f < 2N_c$ there are two solutions: one with $S = 0$ in the baryonic branch and one with $S \neq 0$ in the non-baryonic branch. In multi-cut cases, this applies to each cut by replacing N_c, S with the corresponding $N_{c,i}, S_i$ for the cut. (3.24)

In the next section we will discuss the condition we have used in above prescription. Also by explicit examples, we will demonstrate that when the gauge theory has a solution with closed cuts ($S_i = 0$), one can reproduce its superpotential in matrix model by setting the corresponding glueballs S_i to zero by hand.

4 Two cut model—cubic tree level superpotential

Now, let us move on to $U(N_c)$ theory with cubic tree level superpotential, where we have two cuts. We will demonstrate that for each closed cut we can set $S = 0$ by hand to reproduce the correct gauge theory superpotential using matrix model.

¹¹That the D3-brane wrapping S^3 cannot exist for $N_f < N_c$ can probably be explained along the same line as [33], by showing that those 3-5 strings are fermionic. Also, note that the gauge group here is $U(N_c)$, not $SU(N_c)$ as in [33], hence the “baryon” is charged under the $U(1)$.

¹²This condition will not work for the $N_f = N_{c,i}$ case.

Specifically, we take the tree level superpotential to be

$$W_{\text{tree}} = \text{Tr}[W(\Phi)] - \sum_{I=1}^{N_f} \tilde{Q}_I(\Phi - z_f)Q^I, \quad (4.1)$$

$$W(z) = \frac{g}{3}z^3 + \frac{m}{2}z^2, \quad W'(z) = gz \left(z + \frac{m}{g} \right) \equiv g(z - a_1)(z - a_2).$$

Here we wrote down $W(z)$ in terms of $g_2 = g$, $g_1 = m$ for definiteness, but mostly we will work with the last expression in terms of g , $a_{1,2}$. The general breaking pattern in the pseudo-confining phase is $U(N_c) \rightarrow U(N_{c,1}) \times U(N_{c,2})$, $N_{c,1} + N_{c,2} = N_c$, $N_{c,i} > 0$. In the quantum theory, the critical points at a_1 and a_2 blow up into cuts along the intervals $[a_1^-, a_1^+]$ and $[a_2^-, a_2^+]$, respectively. Namely, we end up with the matrix model curve (2.7), which in this case is

$$y_m^2 = W'(z)^2 + f_1(z) = g^2(z - a_1^-)(z - a_1^+)(z - a_2^-)(z - a_2^+). \quad (4.2)$$

We will call the cuts along $[a_1^-, a_1^+]$ and $[a_2^-, a_2^+]$ respectively the “first cut” and the “second cut” henceforth. One important difference from the quadratic case is that, we can study a process where $N_f \geq 2N_{c,i}$ flavor poles are near the i -th cut in the cubic case.

As we have mentioned, our concern is whether the cut is closed or not. Also from the experiences in the factorization it can be seen that for $N_f > N_{c,i}$, when closed cut is produced, the closed cut and the poles are on top of each other¹³. With all these considerations we take the following condition to constrain the position of the poles:¹⁴

$$z_f = a_1^-. \quad (4.3)$$

If there are $S_1 = 0$ solutions in which the closed cut and the poles are on top of each other, then all such solutions can be found by solving the factorization problem under the constraint (4.3), since for such solutions $z = a_1^- = a_1^+$ obviously. One could impose a further condition $S_1 = 0$, or equivalently $a_1^- = a_1^+$ if one wants just closed cut solutions, but we would like to know that there also are solutions with $S_1 \neq 0$ for $N_f < 2N_{c,1}$, so we do not do that.

To summarize, what we are going to do below is: first we explicitly solve the factorization problem under the constraint (4.3), and confirm that the $S_1 = 0$ solution exists when $N_f > N_{c,i}$. Then, we reproduce the gauge theory superpotential in matrix model by setting $S_1 = 0$ by hand.

¹³We do not discuss the $N_f = N_{c,i}$ case where closed cut and poles are not at the same point. However because the $S = 0$ solution in this case is an on-shell solution, we can reproduce the gauge theory result in matrix model without setting $S = 0$ by hand.

¹⁴We could choose $z_f = a_1^+$ or $z_f = a_2^\pm$ instead of (4.3), but the result should be all the same, so we take (4.3) without loss of generality.

Before plunging into that, we must discuss one aspect of the constraint (4.3) and the r -branches, in order to understand the result of the factorization method. If one solves the factorization equation for a given flavor mass $m_f = -z_f$ (without imposing the constraint (4.3)), then in general one will find multiple r -branches labeled by an integer K with range $0 \leq K \leq \min(N_c, [\frac{N_f}{2}])$ (see Eq. (2.17)). This is related to the fact that the factorization method cannot distinguish between the poles on the first sheet and the ones on the second sheet. The r -branch labeled by K corresponds to distributing $N_f - K$ poles on the second sheet and K poles on the first sheet. We are not interested in such configurations; we want to put N_f poles at the same point on the same sheet. However, as we discuss now, we actually do not have to worry about the r -branches under the constraint (4.3).

The r -branches with different K are different vacua in general. However, under the constraint (4.3) these r -branches become all identical because at the branch point $z = a_i^\pm$ there is no distinction between the first and second sheets. This can be easily seen in the matrix model approach. From the equation (2.12), the effective glueball superpotential for the two cut model with $N_f - K$ poles at \tilde{q}_f on the second sheet and K poles at q_f on the first sheet is

$$W_{\text{eff}} = -\frac{1}{2}(N_{c,1}\Pi_1 + N_{c,2}\Pi_2) - \frac{1}{2}(N_f - K)\Pi_f^{(2)} - \frac{1}{2}K\Pi_f^{(1)} + \frac{1}{2}(2N_c - N_f)W(\Lambda_0) \\ + \frac{1}{2}N_f W(q) - \pi i(2N_c - N_f)S + 2\pi i\tau_0 S + 2\pi i b_1 S_1,$$

where the periods are defined by

$$S_i = \frac{1}{2\pi i} \int_{A_i} R(z) dz, \quad \Pi_i = 2 \int_{a_i^-}^{\Lambda_0} y(z) dz, \\ \Pi_f^{(2)} = \int_{\tilde{q}_f}^{\tilde{\Lambda}_0} y(z) dz, \quad \Pi_f^{(1)} = \int_{q_f}^{\tilde{\Lambda}_0} y(z) dz = \left[\int_{q_f}^{\tilde{q}_f} + \int_{\tilde{q}_f}^{\tilde{\Lambda}_0} \right] y(z) dz \equiv \Delta\Pi_f + \Pi_f^{(2)}$$

with $i = 1, 2$. The periods $\Pi_f^{(1)}$, $\Pi_f^{(2)}$ are associated with the poles on the first sheet and the ones on the second sheet, respectively. The contour C_2 for $\Pi_f^{(2)}$ is totally on the second sheet, while the contour C_1 for $\Pi_f^{(1)}$ is from q_f on the first sheet, through a cut, to $\tilde{\Lambda}_0$ on the second sheet. These contours are shown in Fig. 5. This r -branch with K poles on the first sheet can be reached by first starting from the pseudo-confining phase with all N_f poles at \tilde{q}_f ($K = 0$) and then moving K poles through the cut to q_f . The path along which the poles are moved in this process is the difference in the contours, $C_1 - C_2 \equiv \Delta C$ ¹⁵.

¹⁵There is ambiguity in taking ΔC ; for example we can take ΔC to go around a_1^+ in Fig. 5. However, the difference in $\int_{\Delta C} y(z) dz$ for such different choices of ΔC is $2\pi i n S_1$, $n \in \mathbb{Z}$, which can be absorbed in redefinition of the theta angle and is immaterial.

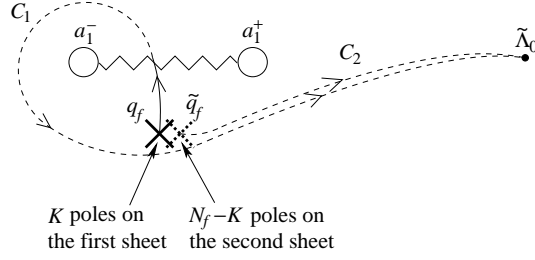


Figure 5: Contours C_1 and C_2 defining $\Pi_f^{(1)}$ and $\Pi_f^{(2)}$, respectively. The part of a contour on the first sheet is drawn in a solid line, while the part on the second sheet is drawn in a dashed line. The $N_f - K$ poles on the second sheet and the K poles on the first sheet are actually on top of each other (more precisely, their projections to the z -plane are.)

When we impose the constraint (4.3), then the difference ΔC vanishes (Fig. 6). Therefore there is no distinction between C_1 , C_2 and hence $\Pi_f^{(1)} = \Pi_f^{(2)}$ for any K . In other words, all K -th branches collapse¹⁶ to the same branch under the constraint (4.3).

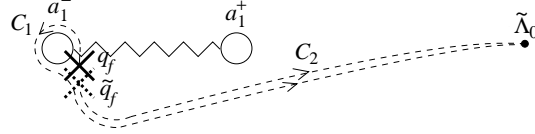


Figure 6: Under the constraint (4.3), contours C_1 and C_2 become degenerate: $C_1 = C_2$.

Now, let us explicitly solve the factorization problem under the constraint (4.3), and check that the $S = 0$ solutions exist as advertised before. In solving the factorization problem, we do not have to worry about the r -branches because there is no distinction among them under the constraint (4.3). Then, we compute the exact superpotential using the data from the factorization and reproduce it in matrix model by setting $S = 0$ by hand when $S = 0$ on the gauge theory side.

For simplicity and definiteness, we consider the case of $U(3)$ gauge group henceforth. We consider $N_f < 2N_c$ flavors, namely $1 \leq N_f \leq 5$, because $N_f \geq 2N_c$ cases are not asymptotically free and cannot be treated in the framework of the Seiberg–Witten theory.

4.1 Gauge theory computation of superpotential

In this subsection, we solve the factorization equation under the constraint (4.3) and compute the exact superpotential, for the system (4.1) with $U(3)$ gauge group and with

¹⁶In fact this collapse was observed in [14, 15] for $SO(N_c)$ and $USp(2N_c)$ gauge groups with massless flavors. We have seen that there are only two branches, i.e., Special branch and Chebyshev branch, which correspond to the baryonic branch and the non-baryonic branch in $U(N_c)$ case.

various breaking patterns.

4.1.1 Setup

The factorization equation for $U(3)$ theory with N_f flavors with mass $m_f = -z_f$ is given by¹⁷ [13]

$$\begin{aligned}\tilde{P}_3(\tilde{z})^2 - 4\Lambda^{6-N_f}(\tilde{z} - z_f)^{N_f} &= \tilde{H}_1(\tilde{z})^2 [W'(\tilde{z})^2 + f_1(\tilde{z})] \\ &= \tilde{H}_1(\tilde{z})^2 (\tilde{z} - a_1^-)(\tilde{z} - a_1^+)(\tilde{z} - a_2^-)(\tilde{z} - a_2^+),\end{aligned}\quad (4.4)$$

where we set $g = 1$ for simplicity and $W'(\tilde{z})$ is given by (4.1). The breaking pattern is assumed to be $U(3) \rightarrow U(N_{c,1}) \times U(N_{c,2})$ with $N_{c,i} > 0$. Here we used new notations to clarify the shift of the coordinate below. For quantities after the shift, we use letters without tildes. Enforcing the constraint (4.3) and shifting \tilde{z} as $\tilde{z} = z + a_1^-$, we can rewrite this relation as follows:

$$\begin{aligned}P_3(z)^2 - 4\Lambda^{6-N_f}z^{N_f} &= H_1(z)^2 [\widetilde{W}'(z)^2 + \tilde{f}_1(z)] = H_1(z)^2 [z(z - \tilde{a}_1^+)(z - \tilde{a}_2^-)(z - \tilde{a}_2^+)] \\ &\equiv H_1(z)^2 [z(z^3 + Bz^2 + Cz + D)].\end{aligned}\quad (4.5)$$

Because of the shift, the polynomials $P_3(z)$ and $H_1(z)$ are different in form from $\tilde{P}_3(\tilde{z})$ and $\tilde{H}_1(\tilde{z})$ in (4.4). We parametrize the polynomials $P_3(z)$ and $H_1(z)$ as

$$P_3(z) = z^3 + az^2 + bz + c, \quad H_1(z) = z - A. \quad (4.6)$$

The parameters B, C can be written in terms of the parameters in

$$\widetilde{W}'(z) = W'(\tilde{z}) = \tilde{z}(\tilde{z} + m_A) = (z + a_1^-)(z + a_1^- + m_A) \equiv (z - a_1)(z - a_2)$$

by comparing the coefficients in (4.2):

$$\begin{aligned}a_1 &= \frac{-B \mp \sqrt{3B^2 - 8C}}{4}, \quad a_2 = \frac{-B \pm \sqrt{3B^2 - 8C}}{4}, \\ \Delta^2 &\equiv (\tilde{a}_2 - \tilde{a}_1)^2 = (a_2 - a_1)^2 = \frac{3B^2 - 8C}{4}.\end{aligned}\quad (4.7)$$

The ambiguity in signs in front of the square roots can be fixed by assuming $a_1 < a_2$. Finally we undo the shift by noting that

$$W(\tilde{z}) = \frac{1}{3}\tilde{z}^3 + \frac{m_A}{2}\tilde{z}^2 = \frac{z^3}{3} - (a_1 + a_2)\frac{z^2}{2} + a_1a_2z + \frac{1}{6}(a_1^3 - 3a_1^2a_2). \quad (4.8)$$

¹⁷From the result of Appendix (A.3), the matrix model curve with flavors does not change even for $N_f > N_c$, contrary to [7].

N_f	breaking pattern $\widehat{U(N_{c,1})} \times U(N_{c,2})$	$N_f \leq N_{c,1}$	$N_{c,1} < N_f < 2N_{c,1}$	$2N_{c,1} \leq N_f$	the first cut is
1	$\widehat{U(2)} \times U(1)$	○	-	-	open
	$\widehat{U(1)} \times U(2)$	○	-	-	open
2	$\widehat{U(1)} \times U(2)$	-	-	○	closed
	$\widehat{U(2)} \times U(1)$	○	-	-	open
3	$\widehat{U(1)} \times U(2)$	-	-	○	closed
	$\widehat{U(2)} \times U(1)$	-	○	-	closed
	$\widehat{U(2)} \times U(1)$	-	○	-	open
4	$\widehat{U(1)} \times U(2)$	-	-	○	closed
	$\widehat{U(2)} \times U(1)$	-	-	○	closed
5	$\widehat{U(1)} \times U(2)$	-	-	○	closed
	$\widehat{U(2)} \times U(1)$	-	-	○	closed

Table 1: The result of factorization of curves for $U(3)$ with up to $N_f = 5$ flavors. “○” denotes which inequality N_f and $N_{c,1}$ satisfy.

With all this setup, we can compute the superpotential as follows. First we factorize the curve according to (4.5). Then we find the Casimirs U_1, U_2, U_3 from $P_3(z)$ ¹⁸ and solve for a_1, a_2 using the last equation of (4.7). Finally we put all these quantities into (4.8) to get the effective action as

$$W_{\text{low}} = \langle \text{Tr } W(\Phi) \rangle = U_3 - (a_1 + a_2)U_2 + a_1 a_2 U_1 + \frac{N_c}{6}(a_1^3 - 3a_1^2 a_2). \quad (4.9)$$

Here, for a_1, a_2 , one can use the first two equations of (4.7). We also want to know whether the first cut (the one along the interval $[a_1^-, a_1^+]$) is closed or not; we expect that the cut closes if we try to bring too many poles near the cut. As is obvious from (4.5), this can be seen from the value of D . If $D = 0$, the cut is closed, while if $D \neq 0$, the cut is open.

4.1.2 The result of factorization problem

We explicitly solved the factorization problem for $U(3)$ gauge theories with $N_f = 1, 2, \dots, 5$ and summarized the result in Table 1. Let us explain about the table. $\widehat{U(N_{c,1})} \times U(N_{c,2})$ denotes the breaking pattern of the $U(3)$ gauge group. The hat on the first factor means that

¹⁸From the coefficients of $P_3(z)$, namely a, b and c , one can compute the Casimirs $U_k = \frac{1}{k} \langle \text{Tr } \Phi^k \rangle$ using the quantum modified Newton relation Appendix (A.1) as explained in Appendix C.

the pole is at one of the branch points of the first cut (Eq. (4.3)) which is associated with the first factor $U(N_{c,1})$. Of course this choice is arbitrary and we may as well choose $U(N_{c,2})$, ending up with the same result. Finally, whether the cut is closed or not depends on whether $D = 0$ or not, as explained below (4.9).

Now let us look carefully at Table 1, comparing it with the prescription (3.24) based on the analysis of the one cut model.

First of all, for $N_f \geq 2N_{c,1}$, the first cut is always closed. We will see below that in these cases with a closed cut the superpotential can be reproduced by setting $S_1 = 0$ by hand in the corresponding matrix model, confirming the prescription (3.24) for $N_f > 2N_{c,1}$.

Secondly, the lines for $N_f = 3$ and $\widehat{U(2)} \times U(1)$ correspond to the $N_{c,1} < N_f < 2N_{c,1}$ part of the prescription (3.24). There indeed are both an open cut solution and a closed cut solution. We will see below that the superpotential of the closed cut solution can be reproduced by setting $S_1 = 0$ by hand in the corresponding matrix model. On the other hand, the superpotential of the open cut solution can be reproduced by not setting $S_1 = 0$, namely by treating S_1 a dynamical variable and extremizing W_{eff} with respect to it. In fact these two solutions are baryonic branch for a closed cut and non-baryonic branch for an open cut.

Finally, for $N_f \leq N_{c,1}$, the cut is always open, which is also consistent with the prescription (3.24). In this case, the superpotential of the open cut solution can be reproduced by extremizing W_{eff} with respect to it, as we will see below. For $N_f = N_{c,1}$ there should be an $S_1 = 0$ ($a_1^- = a_1^+$) solution for some z_f (corresponding to $U(2)$ theory with $N_f = 2$ in the $r = 0$ branch) in the quadratic case, but under the constraint (4.3) we cannot obtain that solution.

Below we present resulting exact superpotentials, for all possible breaking patterns. For simplicity, we do not take care of phase factor of Λ which gives rise to the whole number of vacua. For details of the calculation, see Appendix C.

Results

Definitions: $W_{\text{cl}} = -\frac{1}{3}$ for $\widehat{U(1)} \times U(2)$ and $W_{\text{cl}} = -\frac{1}{6}$ for $\widehat{U(2)} \times U(1)$. For simplicity we set $g = 1$ and $\Delta = a_2 - a_1 = -m/g = 1$.

• $\widehat{U(1)} \times U(2)$ with $N_f = 1$

$$W_{\text{low}} = W_{\text{cl}} - 2T - \frac{5T^2}{2} + \frac{115T^3}{12} - \frac{245T^4}{4} + \frac{30501T^5}{64} - \frac{12349T^6}{3} + \dots, \quad T \equiv \Lambda^{\frac{5}{2}}.$$

- $\widehat{\mathbf{U}(2)} \times \mathbf{U}(1)$ with $\mathbf{N_f} = 1$

$$W_{\text{low}} = W_{\text{cl}} - \frac{5T^2}{2} + \frac{5T^3}{3} - \frac{11T^4}{3} + 11T^5 - \frac{235T^6}{6} + \dots, \quad T \equiv \Lambda^{\frac{5}{3}}.$$

- $\widehat{\mathbf{U}(1)} \times \mathbf{U}(2)$ with $\mathbf{N_f} = 2$

$$W_{\text{low}} = W_{\text{cl}} + 2T^2 - 6T^4 - \frac{32T^6}{3} - 40T^8 - 192T^{10} - \frac{3136T^{12}}{3} + \dots, \quad T \equiv \Lambda.$$

- $\widehat{\mathbf{U}(2)} \times \mathbf{U}(1)$ with $\mathbf{N_f} = 2$

$$W_{\text{low}} = W_{\text{cl}} - 2T^4 - \frac{16T^6}{3} - 24T^8 - 128T^{10} - \frac{2240T^{12}}{3} + \dots, \quad T \equiv \Lambda.$$

- $\widehat{\mathbf{U}(1)} \times \mathbf{U}(2)$ with $\mathbf{N_f} = 3$

$$W_{\text{low}} = W_{\text{cl}} + 2T - \frac{19T^2}{2} + \frac{51T^3}{4} + \frac{157T^4}{4} + \frac{5619T^5}{64} + \frac{33T^6}{2} + \dots, \quad T \equiv \Lambda^{\frac{3}{2}}.$$

- $\widehat{\mathbf{U}(2)} \times \mathbf{U}(1)$ with $\mathbf{N_f} = 3$: two solutions

$$W_{\text{low,baryonic}} = W_{\text{cl}} + T - \frac{5T^2}{2} - 33T^3 - 543T^4 - 10019T^5 - \frac{396591T^6}{2} + \dots, \quad T \equiv \Lambda^3.$$

$$W_{\text{low}} = W_{\text{cl}} + \Lambda^3.$$

- $\widehat{\mathbf{U}(1)} \times \mathbf{U}(2)$ with $\mathbf{N_f} = 4$

$$W_{\text{low}} = W_{\text{cl}} + 2T - 13T^2 + \frac{176T^3}{3} - 138T^4 + 792T^6 - 9288T^8 + 137376T^{10} - 2286144T^{12} + \dots, \quad T \equiv \Lambda.$$

- $\widehat{\mathbf{U}(2)} \times \mathbf{U}(1)$ with $\mathbf{N_f} = 4$

$$W_{\text{low}} = W_{\text{cl}} + T - 6T^2 - \frac{40T^3}{3} - 56T^4 - 288T^5 - \frac{4928T^6}{3} - 9984T^7 - 63360T^8 - \frac{1244672T^9}{3} - 2782208T^{10} - 19009536T^{11} \dots, \quad T \equiv \Lambda^2.$$

- $\widehat{\mathbf{U}(1)} \times \mathbf{U}(2)$ with $\mathbf{N_f} = 5$

$$W_{\text{low}} = W_{\text{cl}} - 2T - \frac{33T^2}{2} - \frac{1525T^3}{12} - \frac{3387T^4}{4} - \frac{314955T^5}{64} - \frac{74767T^6}{3} + \dots, \quad T \equiv \Lambda^{\frac{1}{2}}.$$

- $\widehat{\mathbf{U}(2)} \times \mathbf{U}(1)$ with $\mathbf{N_f} = 5$

$$W_{\text{low}} = W_{\text{cl}} + T - \frac{19T^2}{2} + \frac{154T^3}{3} - 132T^4 + 828T^6 + \dots, \quad T \equiv \Lambda.$$

4.2 Matrix model computation of superpotential

In this subsection we compute the superpotential of the system (4.1) in the framework of [1]. If all the N_f flavors have the same mass $m_f = -z_f$, the effective glueball superpotential $W_{\text{eff}}(S_j)$ for the pseudo-confining phase with breaking pattern $U(N_c) \rightarrow \prod_{i=1}^n U(N_{c,i})$, $\sum_{i=1}^n N_{c,i} = N_c$ is, from (2.12),

$$W_{\text{eff}}(S_j) = -\frac{1}{2} \sum_{i=1}^n N_{c,i} \Pi_i - \frac{N_f}{2} \Pi_f + \left(N_c - \frac{N_f}{2}\right) W(\Lambda_0) + \frac{N_f}{2} W(z_f) - 2\pi i \left(N_c - \frac{N_f}{2}\right) S + 2\pi i \tau_0 S + 2\pi i \sum_{i=1}^{n-1} b_i S_i, \quad (4.10)$$

where the periods associated with adjoint and fundamentals are defined by

$$\Pi_i(S_j) \equiv 2 \int_{a_i^-}^{\Lambda_0} y(z) dz, \quad \Pi_f(S_j) \equiv \int_{\tilde{z}_f}^{\tilde{\Lambda}_0} y(z) dz = - \int_{z_f}^{\Lambda_0} y(z) dz, \\ y(z) = \sqrt{W'(z)^2 + f_1(z)}.$$

For cubic tree level superpotential (4.1), the periods $\Pi_{1,2}(S_j)$ were computed by explicitly evaluating the period integrals by power expansion in [31], as

$$\begin{aligned} \frac{\Pi_1}{2g\Delta^3} &= \frac{1}{g\Delta^3} [W(\Lambda_0) - W(a_1)] + s_1 \left[1 + \log \left(\frac{\lambda_0^2}{s_1} \right) \right] + 2s_2 \log \lambda_0 \\ &\quad + (-2s_1^2 + 10s_1s_2 - 5s_2^2) + \left(-\frac{32}{3}s_1^3 + 91s_1^2s_2 - 118s_1s_2^2 + \frac{91}{3}s_2^3 \right) \\ &\quad + \left(-\frac{280}{3}s_1^4 + \frac{3484}{3}s_1^3s_2 - 2636s_1^2s_2^2 + \frac{5272}{3}s_1s_2^3 - \frac{871}{3}s_2^4 \right) + \dots, \\ \frac{\Pi_2}{2g\Delta^3} &= \frac{1}{g\Delta^3} [W(\Lambda_0) - W(a_2)] + s_2 \left[1 + \log \left(\frac{\lambda_0^2}{-s_2} \right) \right] + 2s_1 \log \lambda_0 \\ &\quad + (2s_2^2 - 10s_1s_2 + 5s_1^2) + \left(-\frac{32}{3}s_2^3 + 91s_1s_2^2 - 118s_1^2s_2 + \frac{91}{3}s_1^3 \right) \\ &\quad + \left(\frac{280}{3}s_2^4 - \frac{3484}{3}s_1s_2^3 + 2636s_1^2s_2^2 - \frac{5272}{3}s_1^3s_2 + \frac{871}{3}s_1^4 \right) + \dots, \end{aligned} \quad (4.11)$$

where $\Delta \equiv a_2 - a_1$, $s_i \equiv S_i/g\Delta^3$, and $\lambda_0 \equiv \Lambda_0/\Delta$.

Under the constraint (4.3), the contours defining Π_1 and Π_f coincide, so

$$\frac{1}{2} \Pi_1 = -\Pi_f = \int_{z_f=a_1^-}^{\Lambda_0} y(z) dz. \quad (4.12)$$

Using this, we can rewrite (4.10) as

$$\begin{aligned}
W_{\text{eff}}(S_1, S_2) = & \left[N_{c,1}W(a_1) + N_{c,2}W(a_2) \right] - \frac{N_f}{2} \left[W(a_1) - W(z_f) \right] \\
& - \tilde{N}_{c,1} \left[\frac{1}{2}\Pi_1 - W(\Lambda_0) + W(a_1) \right] - N_{c,2} \left[\frac{1}{2}\Pi_2 - W(\Lambda_0) + W(a_2) \right] \\
& - 2\pi i(\tilde{N}_{c,1} + N_{c,2})S + 2\pi i\tau_0 S + 2\pi i b_1 S_1.
\end{aligned} \tag{4.13}$$

Here we rearranged the terms taking into account the fact that the periods take the form $\frac{1}{2}\Pi_i = W(\Lambda_0) - W(a_i) + (\text{quantum correction of order } \mathcal{O}(S_i))$, and also the fact that we are considering $z_f = a_1^- \simeq a_1$ (thus the second term). The first line corresponds to the classical contribution, while the second and third lines correspond to quantum correction. Furthermore, we defined $\tilde{N}_{c,1} \equiv N_{c,1} - N_f/2$.

We would like to extremize this W_{eff} (4.13) with respect to $S_{1,2}$, and compute the low energy superpotential that can be compared with the W_{low} obtained in the previous subsection using gauge theory methods. In doing that, one should be careful to the fact that one should treat the mass z_f as an external parameter which is independent of $S_{1,2}$ although we are imposing the constraint (4.3), $z_f = a_1^- = a_1^-(S_1, S_2)$. Where is the z_f dependence in (4.13)? Firstly, z_f appears explicitly in the second term in (4.13). Therefore, when we differentiate W_{eff} with respect to $S_{1,2}$, we should exclude this term. Secondly, there is a more implicit dependence on z_f in $\Pi_f = -\int_{z_f}^{\Lambda_0} y(z)dz$, which we replaced with $-\Pi_1/2$ using (4.12). If we forget to treat z_f as independent of S_i , then we get an apparently unwanted, extra contribution as $\frac{\partial \Pi_f}{\partial S_i} = -\int_{z_f}^{\Lambda_0} \frac{\partial y(z)}{\partial S_i} dz - \frac{\partial z_f}{\partial S_i} \cdot y(z)|_{z=z_f}$. However, this last term actually does not make difference because

$$y(z)|_{z=z_f} = g \sqrt{(z - z_f)(z - a_1^+)(z - a_2^-)(z - a_2^+)} \Big|_{z=z_f} = 0.$$

Therefore what one should do is: i) plug the expression (4.11) into (4.13), ii) solve the equation of motion for $S_{1,2}$ using (4.13) without the second term, and then iii) substitute back the value of $S_{1,2}$ into (4.13), now with the second term included.

Solving the equation of motion can be done by first writing the Veneziano–Yankielowicz term (log and linear terms) as

$$\begin{aligned}
& - \tilde{N}_{c,1} \left[\frac{1}{2}\Pi_1 - W(\Lambda_0) + W(a_1) \right] - N_{c,2} \left[\frac{1}{2}\Pi_2 - W(\Lambda_0) + W(a_2) \right] \\
& \quad - 2\pi i(\tilde{N}_{c,1} + N_{c,2})S + 2\pi i\tau_0 S + 2\pi i b_1 S_1 \\
& = g\Delta^3 \left\{ \tilde{N}_{c,1}s_1[1 - \log(s_1/\lambda_1^3)] + N_{c,2}s_2[1 - \log(s_2/\lambda_2^3)] + \mathcal{O}(s_i^2) \right\},
\end{aligned}$$

where

$$\begin{aligned}\lambda_1^{3\tilde{N}_{c,1}} &= \lambda_0^{2(\tilde{N}_{c,1}+2N_{c,2})} e^{2\pi i(\tilde{N}_{c,1}+N_{c,2})-2\pi i\tau_0-2\pi ib_1}, \\ \lambda_2^{3N_{c,2}} &= (-1)^{N_{c,2}} \lambda_0^{2\tilde{N}_{c,1}+2N_{c,2}} e^{2\pi i(\tilde{N}_{c,1}+N_{c,2})-2\pi i\tau_0},\end{aligned}$$

and then solving the equation of motion perturbatively in $\lambda_{1,2}$. In this way, one can straightforwardly reproduce the results obtained in the previous section in the case with the first cut *open*. In the case with the first cut *closed*, in order to reproduce the results in the previous section, one should first set $S_1 = 0$ by hand, and then extremize W_{eff} with respect to the remaining dynamical variable S_2 .

Following the procedure above, we checked explicitly that extremizing $W_{\text{eff}}(S_1, S_2)$ (open cut) or $W_{\text{eff}}(S_1=0, S_2)$ (closed cut) reproduces the W_{low} up the order presented in the previous section, for all breaking patterns for $U(3)$ theory.

In the above, we concentrated the explicit calculations of effective superpotentials in $U(3)$ theory with cubic tree level superpotential. These explicit examples are useful to see that the prescription (3.24) really works; one can first determine using factorization method when we should set $S_i = 0$ by hand, and then explicitly check that the superpotential obtained by gauge theory can be reproduced by matrix model.

However, if one wants only to show the equality of the two effective superpotentials on the gauge theory and matrix model sides, one can actually prove it in general cases. In Appendix B, we prove this equivalence for $U(N_c)$ gauge theory with an degree $(k+1)$ tree level superpotential where $k+1 < N_c$. There, we show the following: if there are solutions to the factorization problem with some cuts closed, then the superpotential W_{low} of the gauge theory can be reproduced by extremizing the glueball superpotential $W_{\text{eff}}(S_i)$ on the matrix model side, after setting the corresponding glueball fields S_i to zero by hand. Note that, on the matrix side we do not know when we should set S_i to zero *a priori*; we can always set S_i to zero in matrix model, but that does not necessarily correspond to a physical solution on the gauge theory side that solves the factorization constraint.

5 Conclusion and some remarks

In this paper, taking $\mathcal{N} = 1$ $U(N_c)$ gauge theory with an adjoint and flavors, we studied the on-shell process of passing N_f flavor poles on top of each other on the second sheet through a cut onto the first sheet. This corresponds to a continuous transition from the pseudo-confining phase with $U(N_c)$ unbroken to the Higgs phase with $U(N_c - N_f)$ unbroken (we are focusing

on one cut). We confirmed the conjecture of [1] that for $N_f < N_c$ the poles can go all the way to infinity on the first sheet, while for $N_f \geq N_c$ there is obstruction. There are two types of obstructions: the first one is that the cut rotates, catches poles and sends them back to the first sheet, while the second one is that the cut closes up before poles reach it. The first obstruction occurs for $N_c \leq N_f < 2N_c$ whereas the second one occurs for $N_c < N_f$.

If a cut closes up, the corresponding glueball S vanishes, which means that the $U(N_c)$ group is completely broken down. This can happen only by condensation of a charged massless degree of freedom, which is missing in the matrix model description of the system. With a massless degree of freedom missing in the description, the $S = 0$ solution should be singular in matrix model in some sense. Indeed, we found that the $S = 0$ solution of the gauge theory does not satisfy the equation of motion in matrix model (with an exception of the $N_f = N_c$ case, where the $S = 0$ solution does satisfy the equation of motion). How to cure this defect of matrix model is simple — the only thing the missing massless degree of freedom does is to make $S = 0$ a solution, so we just set $S = 0$ by hand in matrix model. We gave a precise prescription (3.24) when we should do this, i.e., *in the baryonic branch for $N_{c,i} \leq N_f < 2N_{c,i}$ and in the $r = N_{c,i}$ non-baryonic branch*, and checked it with specific examples.

The string theory origin of the massless degree of freedom can be conjectured by generalizing the argument in [3]. We argued that it should be the D3-brane wrapping the blown up S^3 , along with fundamental strings emanating from it and ending on the noncompact D5-branes in the Calabi–Yau geometry.

Although we checked that the prescription works, the string theory picture of the $S = 0$ solution needs further refinement, which we leave for future research. For example, although we argued that some extra degree of freedom makes $S = 0$ a solution, we do not have the precise form of the superpotential including that extra field. It is desirable to derive it and show that $S = 0$ is indeed a solution, as was done in [3] in the case without flavors. Furthermore, we saw that there is an on-shell $S = 0$ solution for $N_f = N_c$. Although this solution solves the equation of motion in matrix model, there should be a massless field behind the scene. It is interesting to look for the nature of this degree of freedom. It cannot be the D3-branes with fundamental strings emanating from it, since for this solution the noncompact D5-branes are at finite distance from the collapsed S^3 and the 3-5 strings are massive. Finally, we found that the $S = 0$ solution is in the baryonic branch. It would be interesting to ask if one can describe the baryonic branch in the matrix model framework by adding some extra degrees of freedom.

In the following, we study some aspects of the theory, which we could not discuss so far. We will discuss generalization to $SO(N_c)$ and $USp(2N_c)$ gauge groups by computing the

effective superpotentials with quadratic tree level superpotential.

5.1 $SO(N_c)$ theory with flavors

Here we consider the one cut model for $SO(N_c)$ gauge theory with N_f flavors. The tree level superpotential of the theory is obtained from $\mathcal{N} = 2$ SQCD by adding the mass m_A for the adjoint scalar Φ

$$W_{\text{tree}} = \frac{m_A}{2} \text{Tr } \Phi^2 + Q^f \Phi Q^{f'} J_{ff'} + Q^f \tilde{m}_{ff'} Q^{f'}. \quad (5.1)$$

where $f = 1, 2, \dots, 2N_f$ and the symplectic metric $J_{ff'}$ and mass matrix for quark $\tilde{m}_{ff'}$ are given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathbf{I}_{N_f \times N_f}, \quad \tilde{m} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \text{diag}(m_1, \dots, m_{N_f}).$$

For this simple case the matrix model curve is given by

$$y(z)^2 = m_A^2 (z^2 - 4\mu^2). \quad (5.2)$$

This Riemann surface is a double cover of the complex z -plane branched at the roots of y_m^2 (that is $z = \pm 2\mu$).

The effective superpotential receives contributions from both the sphere and the disk amplitudes in the matrix model [6] and the explicit form was given in [1] for $U(N_c)$ gauge theory with flavors. Now we apply this procedure to our $SO(N_c)$ gauge theory with flavors and it turns out the following expression

$$\begin{aligned} W_{\text{eff}} = & -\frac{1}{2} \left(\sum_{i=-n}^n N_{c,i} - 2 \right) \int_{\hat{B}_i^r} y(z) dz - \frac{1}{4} \sum_{I=1}^{2N_f} \int_{\tilde{q}_I}^{\tilde{\Lambda}_0} y(z) dz + \frac{1}{2} (2N_c - 4 - 2N_f) W(\Lambda_0) \\ & + \frac{1}{2} \sum_{I=1}^{2N_f} W(z_I) - \pi i (2N_c - 4 - 2N_f) S + 2\pi i \tau_0 S + 2\pi i \sum_{i=1}^n b_i S_i \end{aligned}$$

where $S = S_0 + 2 \sum_{i=1}^n S_i$ and z_I is the root of

$$B(z) = \det m(z) = \prod_{I=1}^{N_f} (z^2 - z_I^2).$$

Since the curve (5.2) is same as the one (3.1) of $U(N_c)$ gauge theory, we can use the

integral results given there to write down the effective superpotential as

$$\begin{aligned}
W_{\text{eff}} = S & \left[\frac{(N_c - 2)}{2} + \log \left(\frac{2^{\frac{N_c-2}{2}} m_A^{\frac{N_c-2}{2}} \Lambda^{N_c-2-N_f} \det z}{S^{\frac{N_c-2}{2}}} \right) \right] \\
& - S \sum_{I=1, r_I=0}^{N_f} \left[-\log \left(\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{2S}{m_A z_I^2}} \right) + \frac{m_A z_I^2}{2S} \left(\sqrt{1 - \frac{2S}{m_A z_I^2}} - 1 \right) + \frac{1}{2} \right] \\
& - S \sum_{I=1, r_I=1}^{N_f} \left[-\log \left(\frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{2S}{m_A z_I^2}} \right) + \frac{m_A z_I^2}{2S} \left(-\sqrt{1 - \frac{2S}{m_A z_I^2}} - 1 \right) + \frac{1}{2} \right].
\end{aligned}$$

We can solve $M(z)$ and $T(z)$ as did for $U(N_c)$ gauge theory. For simplicity we take all $r_I = 0$, i.e., all poles at the second sheet. For the I -th block diagonal matrix element of $M(z)$ ($I = 1, \dots, N_f$) it is given by

$$M_I(z) = \begin{pmatrix} 0 & -\frac{R(z) - R(q_I = m_I)}{z - m_I} \\ \frac{R(z) - R(q_I = -m_I)}{z + m_I} & 0 \end{pmatrix}$$

where

$$R(z) = m_A \left(z - \sqrt{z^2 - 4\mu^2} \right).$$

Expanding $M_I(z)$ in the series of z we can find $\langle Q^f \Phi Q^{f'} J_{ff'} + Q^f \tilde{m}_{ff'} Q^{f'} \rangle = 2N_f S$.

The gauge invariant operator $T(z)$ can be constructed similarly as follows:

$$T(z) = \frac{B'(z)}{2B(z)} - \sum_{I=1}^{N_f} \frac{y(q_I) z_I}{y(z) (z^2 - z_I^2)} + \frac{c(z)}{y(z)} - \frac{2R(z)}{z y(z)} \quad (5.3)$$

where

$$c(z) = \left\langle \text{Tr} \frac{W'(z) - W'(\Phi)}{z - \Phi} \right\rangle - \sum_{I=1}^{N_f} \frac{z W'(z) - z_I W'(z_I)}{(z^2 - z_I^2)}.$$

For the theory without the quarks, the Konishi anomaly was derived in [18, 17, 34]. The last term in (5.3) reflects the action of orientifold. For our example we have

$$c(z) = m_A (N_c - N_f).$$

and

$$\begin{aligned}
T(z) = \frac{1}{z} N_c + \frac{1}{z^3} & \left[\sum_{I=1}^{N_f} z_I \left(z_I - \sqrt{z_I^2 - 4\mu^2} \right) + 2\mu^2 (N_c - 2 - N_f) \right] \\
& + \frac{1}{z^5} \left[\sum_{I=1}^{N_f} z_I^4 - \sum_{I=1}^{N_f} \sqrt{z_I^2 - 4\mu^2} z_I (z_I^2 + 2\mu^2) + 6\mu^4 (N_c - N_f) - 12\mu^4 \right] + \mathcal{O} \left(\frac{1}{z^7} \right)
\end{aligned}$$

where for equal mass of flavor, we get $\langle \text{Tr } \Phi^2 \rangle = N_f q \left(q - \sqrt{q^2 - 4\mu^2} \right) + 2\mu^2 (N_c - 2 - N_f)$ and $\langle \text{Tr } \Phi^4 \rangle = N_f q^4 - N_f \sqrt{q^2 - 4\mu^2} q (q^2 + 2\mu^2) + 6\mu^4 (N_c - 2 - N_f)$.

Let us assume the mass of flavors are the same and K of them (in this case, $r_I = 1$) locate at the first sheet while the remainder $(N_f - K)$ where $r_I = 0$ are at the second sheet. Then from the effective superpotential, it is ready to extremize this with respect to the glueball field S ¹⁹

$$0 = \log \left(\frac{\Lambda_1^{\frac{3}{2}(N_c-2)-N_f}}{S^{\frac{N_c-2}{2}}} \right) + K \log \left(\frac{z_f - \sqrt{z_f^2 - \frac{2S}{m_A}}}{2} \right) + (N_f - K) \log \left(\frac{z_f + \sqrt{z_f^2 - \frac{2S}{m_A}}}{2} \right). \quad (5.4)$$

Or by rescaling the fields $\hat{S} = \frac{S}{2m_A\Lambda^2}$, $\hat{z}_f = \frac{z_f}{\Lambda}$ one gets the solution and consider for $K = 0$ case

$$1 = \hat{S}^{-\frac{N_c-2}{2}} \left(\frac{\hat{z}_f + \sqrt{\hat{z}_f^2 - 4\hat{S}}}{2} \right)^{N_f}.$$

This equation is the same as the one in $U(N_c)$ case with $N_c \rightarrow \frac{N_c-2}{2}$, so the discussion of passing poles will go through without modification and the result is when $N_f \geq \frac{N_c-2}{2}$, the on-shell poles at the second plane cannot pass the cut to reach the first sheet far away from the cut.

By using the condition (5.4), one gets the on-shell effective superpotential

$$W_{\text{eff, on-shell}} = \frac{1}{2} (N_c - 2 - N_f) S + \frac{1}{2} m_A z_f^2 \left(N_f + (2K - N_f) \sqrt{1 - \frac{2S}{m_A z_f^2}} \right).$$

It can be checked that this is the same as $\frac{1}{2} m_A \langle \text{Tr } \Phi^2 \rangle$.

5.2 $USp(2N_c)$ theory with flavors

For the $USp(2N_c)$ gauge theory with N_f flavors we will sketch the discussion because most of them is similar to $SO(N_c)$ gauge theory. The tree level superpotential is given by (5.1) but

¹⁹One can easily check that this equation with parameters (N_c, N_f, K) is equivalent to the one with parameters $(N_c - 2r, N_f - 2r, K - r)$. The equation of motion for glueball field is the same. Since the equation of motion for both r -th Higgs branch and $(N_f - r)$ -th Higgs branch is the same, one expects that both branches have some relation. By redefinition of $S \rightarrow \frac{4m_A^2\Lambda^4}{S} \equiv \tilde{S}$, $z_f \rightarrow \frac{2m_A\Lambda^2}{\tilde{S}} z_f \equiv \tilde{z}_f$ we get the final relation between K Higgs branch and $(N_f - K)$ Higgs branch.

with J the symplectic metric, and $\tilde{m}_{ff'}$ the quark mass given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathbf{I}_{N_c \times N_c}, \quad \tilde{m} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \text{diag}(m_1, \dots, m_{N_f}).$$

We parametrize the matrix model curve and the resolvent $R(z)$ as

$$y_m^2 = W'(z)^2 + f(z) = m_A^2 (z^2 + 4\mu^2), \\ R(z) = m_A \left(z - \sqrt{z^2 + 4\mu^2} \right).$$

The effective superpotential is given by

$$W_{\text{eff}} = S(N_c + 1) \left[1 + \log \left(\frac{\tilde{\Lambda}^3}{S} \right) \right] \\ - S \sum_{I=1, r_I=0}^{N_f} \left[-\log \left(\frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{2S}{m_A z_I^2}} \right) - \frac{m_A z_I^2}{2S} \left(\sqrt{1 + \frac{2S}{m_A z_I^2}} - 1 \right) + \frac{1}{2} \right] \\ - S \sum_{I=1, r_I=1}^{N_f} \left[-\log \left(\frac{1}{2} - \frac{1}{2} \sqrt{1 + \frac{2S}{m_A z_I^2}} \right) - \frac{m_A z_I^2}{2S} \left(-\sqrt{1 + \frac{2S}{m_A z_I^2}} - 1 \right) + \frac{1}{2} \right]$$

and $T(z)$ is given by

$$T(z) = \frac{B'(z)}{2B(z)} - \sum_{I=1}^{N_f} \frac{y(q_I) z_I}{y(z) (z^2 - z_I^2)} + \frac{c(z)}{y(z)} + \frac{2R(z)}{z y(z)}.$$

Note the last term (different sign) compared with the $SO(N_c)$ gauge theory. From the solution of $M(z)$, we can show that although $\langle Q^f (\Phi J)_{ff'} Q^{f'} + Q^f \tilde{m}_{ff'} Q^{f'} J \rangle = 2N_f S \neq 0$, we still have on-shell relation $\frac{1}{2} m_A \langle \text{Tr } \Phi^2 \rangle = W_{\text{eff}}$.

The equation of motion is given by ²⁰

$$0 = \log \hat{S}^{-N_c-1} + K \log \left(\frac{\hat{z}_f - \sqrt{\hat{z}_f^2 + 4\hat{S}}}{2} \right) + (N_f - K) \log \left(\frac{\hat{z}_f + \sqrt{\hat{z}_f^2 + 4\hat{S}}}{2} \right)$$

where $\hat{S} = \frac{S}{2m_A \Lambda^2}$, $\hat{z}_f = \frac{z_f}{\Lambda}$. From this we can read out the following result: when $N_f \geq N_c + 1$, the on-shell poles at the second sheet cannot pass through the cut to reach the first sheet far away from the cut.

²⁰One can easily check that this equation with parameters $(2N_c, N_f, K)$ is equivalent to the one with parameters $(2N_c - 2r, N_f - 2r, K - r)$. In other words, the equation of motion for glueball field is the same. Since the equation of motion for both r -th Higgs branch and $(K - r)$ -th Higgs branch is equivalent to each other, one expects that both branches have some relation.

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Appendix

A On matrix model curve with $N_f(> N_c)$ flavors

In this Appendix we prove by strong coupling analysis that matrix model curve corresponding to $U(N_c)$ supersymmetric gauge theory with N_f flavors is exactly the same as the one without flavors when the degree $(k+1)$ of tree level superpotential W_{tree} is less than N_c ²¹. This was first proved in [32] but the derivation was valid only for the range $N_f < N_c$. Then in [13], the proof was extended to the cases with the range $2N_c > N_f \geq N_c$. However, in [13], the characteristic function $P_{N_c}(x)$ was defined by $P_{N_c}(x) = \det(x - \Phi)$, without taking into account the possible quantum corrections due to flavors. In consequence, it appeared that the matrix model curve is changed by addition of flavors. In this Appendix, we use the definition of $P_{N_c}(x)$ proposed in Eq. (C.2) of [1]:

$$P_{N_c}(x) = x^{N_c} \exp \left(- \sum_{i=1}^{\infty} \frac{U_i}{x^i} \right) + \Lambda^{2N_c - N_f} \frac{\widehat{B}(x)}{x^{N_c}} \exp \left(\sum_{i=1}^{\infty} \frac{U_i}{x^i} \right), \quad (\text{A.1})$$

which incorporates quantum corrections and reduces to $P_{N_c}(x) = \det(x - \Phi)$ for $N_f = 0$, and see clearly that the matrix model curve is not changed, even when the number of flavors is more than N_c . Since $P_{N_c}(x)$ is a polynomial in x , (A.1) can be used to express U_r with $r > N_c$ in terms of U_r with $r \leq N_c$ by imposing the vanishing of the negative power terms in x .

Assuming that the unbroken gauge group at low energy is $U(1)^n$ with $n \leq k$, the factorization form of Seiberg–Witten curve can be written as,

$$P_{N_c}^2(x) - 4\Lambda^{2N_c - N_f} \widehat{B}(x) = H_{N_c - n}^2(x) F_{2n}(x).$$

²¹The generalized Konishi anomaly equation of $R(z)$ given in (2.6) is same with or without flavors, so the form of the solution is the same for gauge theory with or without flavor. In this Appendix we use another method to prove this result.

The effective superpotential with this double root constraint can be written as follows ²².

$$W_{\text{eff}} = \sum_{r=0}^k g_r U_{r+1} + \sum_{i=1}^{N_c-n} \left(L_i \oint \frac{P_{N_c}(x) - 2\epsilon_i \Lambda^{N_c - \frac{N_f}{2}} \sqrt{\widehat{B}(x)}}{x - p_i} dx + B_i \oint \frac{P_{N_c}(x) - 2\epsilon_i \Lambda^{N_c - \frac{N_f}{2}} \sqrt{\widehat{B}(x)}}{(x - p_i)^2} dx \right).$$

The equations of motion for B_i and p_i are given as follows respectively:

$$0 = \oint \frac{P_{N_c}(x) - 2\epsilon_i \Lambda^{N_c - \frac{N_f}{2}} \sqrt{\widehat{B}(x)}}{(x - p_i)^2} dx, \quad 0 = 2B_i \oint \frac{P_{N_c}(x) - 2\epsilon_i \Lambda^{N_c - \frac{N_f}{2}} \sqrt{\widehat{B}(x)}}{(x - p_i)^3} dx.$$

Assuming that the factorization form does not have any triple or higher roots, we obtain $B_i = 0$ at the level of equation of motion. Next we consider the equation of motion for U_r :

$$0 = g_{r-1} + \sum_{i=1}^{N_c-n} \oint \left[\frac{P_{N_c}}{x^r} - 2 \frac{x^{N_c}}{x^r} \exp \left(- \sum_{i=1}^{\infty} \frac{U_i}{x^i} \right) \right] \frac{L_i}{x - p_i} dx$$

where we used $B_i = 0$ and (A.1) to evaluate $\frac{\partial P_{N_c}}{\partial U_r}$. Now, as in [8], we multiply this by z^{r-1} and sum over r .

$$W'(z) = - \oint \frac{P_{N_c}}{x - z} \sum_{i=1}^{N_c-n} \frac{L_i}{x - p_i} dx + \oint \frac{2x^{N_c}}{x - z} \exp \left(- \sum_{k=1}^{\infty} \frac{U_k}{x^k} \right) \sum_{i=1}^{N_c-n} \frac{L_i}{x - p_i} dx.$$

Defining the polynomial $Q(x)$ in terms of

$$\sum_{i=1}^{N_c-n} \frac{L_i}{x - p_i} = \frac{Q(x)}{H_{N_c-n}(x)}, \quad (\text{A.2})$$

and also using (A.1) and factorization form, we obtain

$$\begin{aligned} W'(z) &= - \oint \frac{P_{N_c}}{x - z} \frac{Q(x)}{H_{N_c-n}(x)} dx + \oint \frac{P_{N_c}}{x - z} \frac{Q(x)}{H_{N_c-n}(x)} dx + \oint \frac{Q(x) \sqrt{F_{2n}(x)}}{x - z} dx \\ &= \oint \frac{Q(x) \sqrt{F_{2n}(x)}}{x - z} dx. \end{aligned}$$

This is nothing but (2.37) in [8]. Since $W'(z)$ is a polynomial of degree k , the Q should be a polynomial of degree $(k - n)$. Therefore, we conclude that the matrix model curve is not changed by addition of flavors:

$$y_m^2 = F_{2n}(x) Q_{k-n}^2(x) = W'_k(x)^2 + \mathcal{O}(x^{k-1}). \quad (\text{A.3})$$

²²If we want to generalize this proof to more general cases in which $k + 1$ is greater than and equals to N_c , we have to take care more constraints like Appendix A in [8], which should be straightforward.

B Equivalence between W_{low} and $W_{\text{eff}}(\langle S_i \rangle)$ with flavors

In this Appendix we prove the equivalence W_{low} in $U(N_c)$ gauge theory with $W_{\text{eff}}(\langle S_i \rangle)$ in corresponding dual geometry when some of the branch cuts on the Riemann surface are closed and the degree $(k+1)$ of the tree level superpotential W_{tree} is less than N_c . This was first proved in [32], however, the proof was only applicable in the $N_f < N_c$ cases. Especially, the field theory analysis in [32] did not work for $N_c \leq N_f < 2N_c$ cases. Furthermore, as we saw in the main text, for some particular choices of z_I (position of the flavor poles), extra double roots appear in the factorization problem. In section 4, we dealt with $U(3)$ with cubic tree level superpotential and saw the equivalence of two effective superpotentials for such special situations. To include these cases we are interested in the Riemann surface that has some closed branch cuts. Therefore our proof is applicable for $U(N_c)$ gauge theories with W_{tree} of degree $k+1$ ($< N_c$) in which some of branch cuts are closed and number of flavors is in the range $N_c \leq N_f < 2N_c$. In addition, we restrict our discussion to the Coulomb phase.

In the discussion below, we follow the strategy developed by Cachazo and Vafa in [36] and use (A.1) as the definition of $P_{N_c}(x)$. We have only to show the two relations:

$$W_{\text{low}}(g_r, z_I, \Lambda) \Big|_{\Lambda \rightarrow 0} = W_{\text{eff}}(\langle S_i \rangle) \Big|_{\Lambda \rightarrow 0}, \quad (\text{B.1})$$

$$\frac{\partial W_{\text{low}}(g_r, z_I, \Lambda)}{\partial \Lambda} = \frac{\partial W_{\text{eff}}(\langle S_i \rangle)}{\partial \Lambda}, \quad (\text{B.2})$$

the equivalence of two effective superpotentials in the classical limit and that of the derivatives of the superpotentials with respect to Λ .

B.1 Field theory analysis

Let k be the order of W'_{tree} and $n(\leq k)$ be the number of $U(1)$ at low energy. Since we are interested in cases with degenerate branch cuts, let us consider the following factorization form ²³:

$$P_{N_c}(x)^2 - 4\Lambda^{2N_c - N_f} \widehat{B}(x) = F_{2n}(x) \left[Q_{k-n}(x) \widetilde{H}_{N_c-k}(x) \right]^2 \equiv F_{2n}(x) [H_{N_c-n}(x)]^2, \quad (\text{B.3})$$

$$W'(x)^2 + f_{k-1}(x) = F_{2n}(x) Q_{k-n}(x)^2.$$

If k equals to n , all the branch cuts in $F_{2k}(x)$ are open. The low energy effective superpotential

²³In the computation below, we will use relation (A.2) and put $g_{k+1} = 1$.

is given by

$$W_{\text{low}} = \sum_{r=1}^{n+1} g_r U_r + \sum_{i=1}^l \left[L_i \left(P_{N_c}(p_i) - 2\epsilon_i \Lambda^{N_c - \frac{N_f}{2}} \sqrt{\widehat{B}(p_i)} \right) + Q_i \frac{\partial}{\partial p_i} \left(P_{N_c}(p_i) - 2\epsilon_i \Lambda^{N_c - \frac{N_f}{2}} \sqrt{\widehat{B}(p_i)} \right) \right],$$

where $l \equiv N - n$ and $P_{N_c}(x)$ is defined by (C.3) or (C.4) in [1],

$$P_{N_c}(x) = \langle \det(x - \Phi) \rangle + \left[\Lambda^{2N_c - N_f} \frac{\widehat{B}(x)}{x^{N_c}} \exp \left(\sum_{i=1}^{\infty} \frac{U_i}{x^i} \right) \right]_+. \quad (\text{B.4})$$

The second term is specific to the $N_f \geq N_c$ case, representing quantum correction. Define

$$K(x) \equiv \left[\Lambda^{2N_c - N_f} \frac{\widehat{B}(x)}{x^{N_c}} \exp \left(\sum_{i=1}^{\infty} \frac{U_i}{x^i} \right) \right]_+.$$

The first term in (B.4) can be represented as $\langle \det(x - \Phi) \rangle \equiv \sum_{k=0}^{N_c} x^{N_c - k} s_k$. The relation between U_i 's and s_k 's are given by the ordinary Newton relation, $ks_k + \sum_{r=1}^k r U_r s_{k-r} = 0$. From the variations of W_{low} with respect to p_i and Q_i , we conclude that $Q_i = 0$ at the level of the equation of motion. In addition, the variation of W_{low} with respect to U_r leads to

$$\begin{aligned} g_r &= - \sum_{i=1}^l L_i \frac{\partial P_{N_c}(p_i)}{\partial U_r} = \sum_{i=1}^l \sum_{j=0}^{N_c} L_i p_i^{N_c - j} s_{j-r} - \sum_{i=1}^l L_i \frac{\partial K(p_i)}{\partial U_r} \\ &= \sum_{i=1}^l \sum_{j=0}^{N_c} L_i p_i^{N_c - j} s_{j-r} - \sum_{i=1}^l L_i \left[\Lambda^{2N_c - N_f} \frac{\widehat{B}(p_i)}{p_i^{N_c + r}} \exp \left(\sum_{k=1}^{\infty} \frac{U_k}{p_i^k} \right) \right]_+. \end{aligned}$$

Let us define

$$\mathcal{G}_r(p_i) \equiv \left[\Lambda^{2N_c - N_f} \frac{\widehat{B}(p_i)}{p_i^{N_c + r}} \exp \left(\sum_{k=1}^{\infty} \frac{U_k}{p_i^k} \right) \right]_+.$$

By using these relations, let us compute W'_{cl}

$$\begin{aligned}
W'_{\text{cl}} &= \sum_{r=1}^{N_c} g_r x^{r-1} \\
&= \sum_{r=-\infty}^{N_c} \sum_{i=1}^l \sum_{j=0}^{N_c} x^{r-1} p_i^{N_c-j} s_{j-r} L_i - \frac{1}{x} \sum_{i=1}^l L_i \det(p_i - \Phi) - \sum_{r=1}^{N_c} \sum_{i=1}^l L_i \mathcal{G}_r(p_i) x^{r-1} \\
&= \sum_{i=1}^l \frac{\det(x - \Phi)}{x - p_i} L_i - \frac{1}{x} \sum_{i=1}^l L_i \det(p_i - \Phi) - \sum_{r=1}^{N_c} \sum_{i=1}^l L_i \mathcal{G}_r(p_i) x^{r-1} \\
&= \sum_{i=1}^l \frac{P_{N_c}(x)}{x - p_i} L_i - \sum_{i=1}^l \frac{K(x)}{x - p_i} L_i - \frac{1}{x} \sum_{i=1}^l L_i P_{N_c}(p_i) + \frac{1}{x} \sum_{i=1}^l L_i K(p_i) \\
&\quad - \sum_{r=1}^{N_c} \sum_{i=1}^l L_i \mathcal{G}_r(p_i) x^{r-1}
\end{aligned} \tag{B.5}$$

where we dropped $\mathcal{O}(x^{-2})$. The fifth term above can be written as

$$- \sum_{r=1}^{N_c} \sum_{i=1}^l L_i \mathcal{G}_r(p_i) x^{r-1} = - \sum_{r=-\infty}^{N_c} \sum_{i=1}^l L_i \mathcal{G}_r(p_i) x^{r-1} + \sum_{i=1}^l \frac{1}{x} L_i K(p_i) + \mathcal{O}(x^{-2}).$$

After some manipulation with the factorization form (B.3), we obtain a relation

$$\begin{aligned}
Q_{k-n}(x) \sqrt{F_{2n}(x)} &= \frac{P_{N_c}(x)}{\tilde{H}_{N_c-k}} - \frac{2\Lambda^{2N_c-N_f} \hat{B}(x)}{\tilde{H}_{N_c-k} x^N} \exp \left(\sum_{i=1}^{\infty} \frac{U_i}{x^i} \right) \\
&= \frac{P_{N_c}(x)}{\tilde{H}_{N_c-k}} - \frac{2K(x)}{\tilde{H}_{N_c-k}} + \mathcal{O}(x^{-2}).
\end{aligned}$$

Substituting this relation into (B.5) we obtain

$$\begin{aligned}
W'_{\text{cl}} &= Q_{k-n}(x) \sqrt{F_{2n}(x)} - \sum_{i=1}^l \frac{1}{x} [L_i P_{N_c}(p_i) - 2L_i K(p_i)] + \sum_{i=1}^l \frac{K(x)}{x - p_i} L_i \\
&\quad - \sum_{r=-\infty}^{N_c} \sum_{i=1}^l L_i \mathcal{G}_r(p_i) x^{r-1} + \mathcal{O}(x^{-2}).
\end{aligned}$$

Finally let us use the following relation ²⁴ ;

$$\sum_{i=1}^l \frac{K(x)}{x - p_i} L_i - \sum_{r=-\infty}^{N_c} \sum_{i=1}^l L_i \mathcal{G}_r(p_i) x^{r-1} = \mathcal{O}(x^{-2}). \tag{B.6}$$

²⁴For simplicity, we ignore $\sum_{i=1}^l L_i$. To prove the relation, let us consider the circle integral over C . Until now, since we assumed $p_i < x$, the point p_i should be included in the contour C . Multiplying $\frac{1}{x^k}$ $k \geq 0$ and taking the circle integral, we can pick up the coefficient c_{k-1} of x^{k-1} . In addition, if we denote a polynomial

After all, by squaring W'_{cl} , we have

$$Q_{k-n}(x)^2 F_{2n}(x) = W_{\text{cl}}'^2(x) + 2 \sum_{i=1}^l \frac{1}{x} [L_i P_{N_c}(p_i) - 2L_i K(p_i)] x^{k-1} + \mathcal{O}(x^{k-2}),$$

$$b_{k-1} = 2 \sum_{i=1}^l \frac{1}{x} [L_i P_{N_c}(p_i) - 2L_i K(p_i)].$$

On the other hand the variation of W_{low} with respect to Λ is given by

$$\frac{\partial W_{\text{low}}}{\partial \log \Lambda^{2N_c - N_f}} = \sum_{i=1}^l L_i K(p_i) - \frac{1}{2} \sum_{i=1}^l L_i P_{N_c}(p_i) = -\frac{b_{k-1}}{4}. \quad (\text{B.7})$$

This is one of the main results for our proof. In the dual geometry analysis below, we will see the similar relation.

In the classical limit, we have only to consider the expectation value of Φ . In our assumption, gauge symmetry breaks as $U(N_c) \rightarrow \prod_i^n U(N_{c,i})$ we have $\text{Tr } \Phi = \sum_{i=1}^n N_{c,i} a_i$. Therefore in the classical limit W_{low} behaves as

$$W_{\text{cl}} = \sum_i^n N_{c,i} W(a_i). \quad (\text{B.8})$$

By comparing (B.7) and (B.8) and the similar result which we will see in the dual geometry analysis below, we will show (B.1) and (B.2). Let us move to the dual geometry analysis.

B.2 Dual geometry analysis with some closed branch cuts

As we have already seen in the main text, a solution with $\langle S_i \rangle = 0$ appears for some special choice of z_I , the position of flavor poles. In our present proof, however, we put some of S_i to be zero from the beginning, *without specifying* z_I . More precisely, what we prove in this

$M \equiv \Lambda^{2N_c - N_f} \frac{\widehat{B}(x)}{x^{N_c}} \exp\left(\sum_{i=1}^{\infty} \frac{U_i}{x^i}\right) \equiv \sum_{j=-\infty}^{\infty} a_j x^j$ we can obtain following relation,

$$\frac{[M]_+}{x^k} = \left[\frac{M}{x^k} \right]_+ + \sum_{j=0}^{k-1} a_j x^{-(k-j)} \iff \frac{K(x)}{x^k} = \mathcal{G}_k(x) + \sum_{j=0}^{k-1} a_j x^{-(k-j)}$$

where right hand side means circle integral of left hand side. Thus, we obtain the c_{k-1} as

$$c_{k-1} = \sum_{j=0}^{k-1} \oint_{x=0} \frac{a_j x^{j-k}}{x - p_i} dx + \sum_{j=0}^{k-1} \oint_{x=p_i} \frac{a_j x^{j-k}}{x - p_i} dx = - \sum_{j=0}^{k-1} \oint_{x=0} a_j x^{j-k} \sum_{n=1}^{\infty} \frac{x^n}{p_i^{n+1}} dx + \sum_{j=0}^{k-1} a_j p_i^{j-k} = 0$$

where we used that around $x = 0$, $\frac{1}{x - p_i} = - \sum_{n=1}^{\infty} \frac{x^n}{p_i^{n+1}}$. Therefore the left hand side of (B.6) can be written as $\sum_{j=-\infty}^{\infty} c_j x^j = \sum_{j=-\infty}^{-2} c_j x^j = \mathcal{O}(x^{-2})$.

Appendix is as follows: for a given choice of z_I , if there exists a solution to the factorization problem with some of $\langle S_i \rangle$ vanishing, then we can construct a dual geometry which gives the same low energy effective superpotential as the one given by the solution to the factorization problem, by setting some of S_i to zero from the beginning. Therefore, this analysis *does not* tell us when a solution with $\langle S_i \rangle = 0$ appears. To know that within the matrix model formalism, we have to go back to string theory and consider an explanation such as the one given in [3].

Now let us start our proof. Again, let k be the degree of tree level superpotential $W'_{\text{tree}}(x)$ and n be the number of $U(1)$ at low energy. To realize this situation, we need to consider that $(k - n)$ branch cuts on the Riemann surface should be closed, which corresponds to $\langle S_i \rangle = 0$. Here, there is one important thing: As we know from the expansion of W_{eff} in terms of Λ (e.g. see (4.13)), we cannot obtain any solutions with $\langle S_i \rangle = 0$ if we assume that S_i is dynamical and solve its equation of motion. Therefore to realize the situation with vanishing $\langle S_i \rangle$, we must put $S_i = 0$ at the off-shell level by hand. With this in mind, let us study dual geometry which corresponds to the gauge theories above. In the field theory, we assumed that the Riemann surface had $(k - n)$ closed branch cuts. Thus, in this dual geometry analysis we must assume that at off-shell level, $(k - n)$ S_i 's must be zero. For convenience, we assume that first n S_i 's are non-zero and the remaining $(k - n)$ vanish,

$$S_i \neq 0, \quad i = 1, \dots, n, \quad S_i = 0, \quad i = n + 1, \dots, k.$$

Therefore the Riemann surface can be written as

$$y^2 = F_{2n}(x)Q_{k-n}^2 = W'(x)^2 + b_{k-1}x^{k-1} + \dots$$

The effective superpotential in dual geometry corresponding to $U(N_c)$ gauge theory with N_f flavors was given in [1] (See also (2.12)) and in the classical limit it behaves as ²⁵

$$W_{\text{eff}}|_{\text{cl}} = \sum_{i=1}^n N_{c,i} W(a_i). \quad (\text{B.9})$$

As discussed in the previous section, existence of flavors does not change the Riemann surface $y(x)$. In other words, Riemann surface is not singular at x_I (roots of $B(x)$),

$$\oint_{x_I} y(x) dx = 0, \quad y(x) = \sqrt{W'(x)^2 + b_{k-1}x^{k-1} + \dots}.$$

²⁵Remember that in this Appendix we are assuming only Coulomb branch. For the Higgs branch, see (7.11) and (7.12) in [1].

Therefore as in [36], by deforming contours of all S_i 's and evaluating the residue at infinity on the first sheet, we obtain the following relation,

$$\sum_{i=1}^n S_i = \sum_{i=1}^k S_i = -\frac{1}{4}b_{k-1},$$

where we used $\sum_{i=n+1}^k S_i = 0$.

With this relation in mind, next we consider the variation of W_{eff} with respect to S_i :

$$\frac{\partial W_{\text{eff}}(S_i, \Lambda)}{\partial S_i} = 0, \quad i = 1 \cdots n. \quad (\text{B.10})$$

Solving these equations, we obtain the expectation values, $\langle S_i \rangle$. Of course, these vacuum expectation values depend on Λ , g_r and z_I . Thus when we evaluate the variation of $W_{\text{eff}}(\langle S_i \rangle, \Lambda)$ with respect to Λ , we have to pay attention to implicit dependence on Λ . However the implicit dependence does not contribute because of the equation of motion (B.10):

$$\begin{aligned} \frac{dW_{\text{eff}}(\langle S_i \rangle, \Lambda)}{d\Lambda} &= \sum_{i=1}^n \frac{\partial \langle S_i \rangle}{\partial \Lambda} \cdot \frac{\partial W_{\text{eff}}(\langle S_i \rangle, \Lambda)}{\partial \langle S_i \rangle} + \frac{\partial W_{\text{eff}}(\langle S_i \rangle, \Lambda)}{\partial \Lambda} \\ &= \frac{\partial W_{\text{eff}}(\langle S_i \rangle, \Lambda)}{\partial \Lambda}. \end{aligned}$$

On the other hand, explicit dependence on Λ can be easily obtained by monodromy analysis. Here let us recall the fact that the presence of fundamentals does not change the Riemann surface. In fact, looking at (2.12) we can read off the dependence from the term $2\pi i \tau_0 = \log \left(\frac{B_L \Lambda^{2N_c - N_f}}{\Lambda_0^{2N_c - L}} \right)$,

$$\frac{dW_{\text{eff}}(\langle S_i \rangle, \Lambda)}{d \log \Lambda^{2N_c - N_f}} = S = -\frac{b_{k-1}}{4}. \quad (\text{B.11})$$

To finish our proof, we have to pay attention to $f_{k-1}(x)$, on-shell. Namely putting $\langle S_i \rangle$ into $f_{k-1}(x)$ what kind of property does it have? To see it, let us consider change of variables from S_i 's to b_i 's. As discussed in [9] the Jacobian of the change is non-singular if $0 \leq j \leq k-2$,

$$\frac{\partial S_i}{\partial b_j} = -\frac{1}{8\pi i} \oint_{A_i} dx \frac{x^j}{\sqrt{W'(x)^2 + f(x)}}.$$

In our present case, since only n of k S_i 's are dynamical variable, we use b_i , $i = 0, \cdots, n-1$ in a function $f_{k-1}(x) = \sum b_i x^i$ as new variables, instead of S_i 's. As discussed in [36, 32, 1], by using Abel's theorem, the equation of motions for b_i 's is interpreted as an existence condition of a meromorphic function that has an N_c -th order pole at infinity on the first sheet and an

$(N_c - N_f)$ -th order zero at infinity on the second sheet of Σ and a first order zero at \tilde{q}_I . For a theory with $N_f \leq 2N_c$, such a function can be constructed as follows [35, 1]:

$$\psi(x) = P_{N_c}(x) + \sqrt{P_{N_c}^2(x) - 4\Lambda^{2N_c - N_f}\widehat{B}(x)}.$$

For this function to be single valued on the matrix model curve $y(x)$, the following condition must be satisfied,

$$\begin{aligned} P_{N_c}^2(x) - 4\Lambda^{2N_c - N_f}\widehat{B}(x) &= F_{2n}(x)H_{N_c - n}^2(x) \\ W'(x)^2 + f(x) &= F_{2n}(x)Q_{k-n}^2(x) \end{aligned}$$

This is exactly the same as the factorization form we already see in the field theory analysis. Therefore the value b_{k-1} of on-shell matrix model curve in dual theory is the same one for field theory analysis. Comparing two results, (B.7) and (B.8) with corresponding results for the dual geometry analysis, (B.9) and (B.11) we have shown the equivalence between these two descriptions of effective superpotentials.

C Computation of superpotential — gauge theory side

In this Appendix, we demonstrate the factorization method used in subsection 4.1 to compute the low energy superpotential, taking the $N_f = 4$ case as an example. Therefore there are two kinds of solutions for the factorization problem (4.5) and (4.6).

• The breaking pattern $\widehat{U(2)} \times U(1)$

The first kind of solution for the factorization problem is given by

$$A = 0, \quad B = 2a, \quad C = a^2 - 4\Lambda^2, \quad D = 0, \quad c = 0, \quad b = 0.$$

In the classical limit $\Lambda \rightarrow 0$, we can see the characteristic function goes as $P_3(x) \rightarrow x^2(x + a)$, which means that the breaking pattern is $\widehat{U(2)} \times U(1)$. Note that since we are assuming $m_f = 0$, the notation “ $\widehat{}$ ” should be used for the gauge group that corresponds to the cut near the critical point at $x = 0$. Inserting these solutions into (4.7) we obtain one constraint,

$$\Delta^2 = a^2 + 8\Lambda^2.$$

We can easily represent a as a Taylor expansion of Λ :

$$a = -1 + 4T + 8T^2 + 32T^3 + 160T^4 + 896T^5 + 5376T^6 + \cdots,$$

where we put $\Delta = 1$ and defined $T \equiv \Lambda^2$.²⁶ The coefficients of $P_3(x)$ are related to the Casimirs $U_j = \frac{1}{j} \langle \text{Tr}[\Phi^j] \rangle$ as follows. For $N_c = 3$, $N_f = 4$, (A.1) reads

$$\begin{aligned} P_3(x) &= x^3 \exp\left(-\sum_{j=1}^{\infty} \frac{U_j}{x^j}\right) + \Lambda^5 \frac{x^4}{x^3} \exp\left(\sum_{j=1}^{\infty} \frac{U_j}{x^j}\right) \\ &= x^3 - U_1 x^2 + \left(-U_2 + \frac{U_1^2}{2} + \Lambda^2\right)x + \left(-U_3 + U_1 U_2 - \frac{U_1^3}{6} - \Lambda^2 U_1\right) + \dots \end{aligned}$$

Comparing the coefficients, we obtain

$$U_1 = -a, \quad U_2 = -b + \frac{a^2}{2} + \Lambda^2, \quad U_3 = -c + ab - \frac{a^3}{3} - a\Lambda^2.$$

Furthermore, one can compute $a_{1,2}$ from (4.7). Plugging all these into (4.9), we finally obtain

$$W_{\text{low}} = W_{\text{cl}} + T - 6T^2 - \frac{40T^3}{3} - 56T^4 - 288T^5 - \frac{4928T^6}{3} + \dots, \quad T \equiv \Lambda^2, \quad W_{\text{cl}} = -\frac{1}{6}.$$

• **The breaking pattern $\widehat{U(1)} \times U(2)$**

The other kind of solution for the factorization problem is given by

$$\begin{aligned} A &= \frac{1}{2}(-a - 2\eta\Lambda), & B &= a - 2\eta\Lambda, & C &= \frac{1}{4}(a + 2\eta\Lambda)^2, \\ D &= 0, & c &= 0, & b &= \frac{1}{4}(a + 2\eta\Lambda)^2 \end{aligned}$$

where $\eta \equiv \pm 1$. These solutions correspond to the breaking pattern $\widehat{U(1)} \times U(2)$ in the classical limit. Inserting these solutions into (4.7) we obtain one constraint,

$$\Delta^2 = \frac{1}{4}(a^2 - 20a\eta\Lambda + 4\Lambda^2).$$

Again, let us represent a as a Taylor series of Λ :

$$a = -2 + 10T - 24T^2 + 144T^4 - 1728T^6 + \dots,$$

where we put $\Delta = 1$, $\eta = 1$ and defined $T \equiv \Lambda$. Doing the same way as previous breaking pattern, we can compute the effective superpotential as

$$W_{\text{low}} = W_{\text{cl}} + 2T - 13T^2 + \frac{176T^3}{3} - 138T^4 + 792T^6 + \dots, \quad T \equiv \Lambda, \quad W_{\text{cl}} = -\frac{1}{3}.$$

The other cases with $N_f = 1, 2, 3$ and 5 can be done analogously.

²⁶If we take care of a phase factor of Λ , we will obtain the effective superpotentials corresponding to each vacuum. However in our present calculation, we want to check whether the effective superpotentials of two method, field theory and dual geometry, agree with each other. Therefore, we have only to pay attention to the coefficients in W_{low} , neglecting the phase factor.

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